CEM 991 **Fall 2021**

Assignment No. 5 (25 pts.) **Due Date: November 15, 2021 (Monday)**

1. (7 pts.) Find the point spectrum (eigenvalues) of the projection operator P_G, where G is the proper subspace of the Hilbert space $\mathcal{H}$ (i.e. $G \subset \mathcal{H}$, $G \neq \mathcal{H}$, $G \neq \{\Theta\}$, where Θ is the zero vector). Find the corresponding eigenspaces (sets of eigenvectors corresponding to eigenvalues of P_G).

2. (9 pts.) The operator U is called *unitary* if (i) its domain is the entire Hilbert space ($D(U) = \mathcal{H}$) (ii) its range is the entire Hilbert space ($\Delta(U) = \mathcal{H}$), and (iii) for every $f, g \in \mathcal{H}$,
$$(Uf, Ug) = (f, g). \tag{1}$$
   (a) Show that $U^{-1} = U^{\dagger}$.

   (b) The above definition of a unitary operator does not mention that unitary operators are also linear operators, and yet it can be shown that they are linear. In particular,
$$U(\alpha f) = \alpha U f, \tag{2}$$
   where α is a scalar (real or complex number). Use the inner product expression $(U(\alpha f) - \alpha U f, U(\alpha f) - \alpha U f)$, the bilinearity and positive-definitiness of the inner product, and the property represented by Eq. (1) to show that Eq. (2) is indeed true.

   (c) Show that eigenvectors f_1 and f_2 belonging to two different eigenvalues λ_1 and λ_2 of a unitary operator U are orthogonal.
   Hint: Consider (Uf_1, Uf_2), use the fact that each eigenvalue of a unitary operator U has an absolute value of 1 [see Problem 3 (b) in Assignment No. 4], and note that a complex number λ, whose absolute value is 1, can always be written as $\lambda = e^{i\alpha}$, where $0 \leq \alpha < 2\pi$.

3. (9 pts.) A linear operator A is called *skew-symmetric* if $A^{\dagger} = -A$.

   (a) Prove that the point spectrum (a set of eigenvalues) of the skew-symmetric operator is a subset of the imaginary axis.

   (b) Prove that the eigenvectors of a skew-symmetric operator A that correspond to two different eigenvalues of A are orthogonal.

   (c) Prove that a differential operator
$$\frac{\partial}{\partial \phi}$$
   is skew-symmetric (ϕ is a polar angle, $\phi \in [0, 2\pi]$).

SOLUTIONS

I. METHOD 1:

Consider the eigenvalue problem,

$$P_G h = \lambda h \quad , \quad h \neq \Theta .$$

In general,

$$h = g + f ,$$

where $g \in G$, $f \in G^{\perp}$.

We obtain,

$$P_G h = P_G g + \underbrace{P_G f}_{\Theta} = g = \lambda h$$

$$= \lambda (g + f) = \lambda g + \lambda f ,$$

so that

$$g = \lambda g + \lambda f$$

or

$$(1-\lambda) g = \lambda f .$$

Now,

$$(1-\lambda) g \in G$$

$$\lambda f \in G^{\perp} .$$

Since the only vector shared by G and $G^\perp$ is the zero vector Θ, we must have

$$\lambda f = \Theta = (1-\lambda) g ,$$

which means that either

- $\lambda = 0$, or
- $\lambda \neq 0$, $f = \Theta$.

$\lambda = 0$

In this case, $(1-\lambda) g = g = \Theta$,

which gives $h = g + f = f \in G^\perp$,

$$P_G h = 0 h , \; h \in G^\perp .$$

$\lambda = 0$ is an eigenvalue, $G^\perp$ is an eigenspace.

$\lambda \neq 0,\ f = \Theta$

In this case,

$$h = g + f = g \in G,$$

$$P_G h = P_G g = g = h = 1 \cdot h,$$

so that $\lambda = 1,\ h \in G.$

$\lambda = 1$ is an eigenvalue, G is the corresponding eigenspace.

The point spectrum consists of two eigenvalues, $\lambda = 0$ and $\lambda = 1$. The corresponding eigenspaces are $G^\perp$ and G, respectively.

METHOD 2:

$$P_G h = \lambda h, \quad h \neq \Theta.$$

We know that $P_G^2 = P_G$, so that

$$P_G^2 h = P_G(\lambda h) = \lambda P_G h = \lambda^2 h$$

and, at the same time,

$$P_G^2 h = P_G h = \lambda h .$$

We obtain,

$$\lambda^2 h = \lambda h, \quad h \neq \Theta ,$$

so that

$$\lambda^2 = \lambda ,$$

which means that $\lambda = 0$ or 1.

The case $\lambda = 0$ gives

$P_G h = \Theta$, which means that $h \in G^\perp$.

The case $\lambda = 1$ gives

$P_G h = h$, which means that $h \in G$.

$\lambda = 0$ and $\lambda = 1$ are eigenvalues, $G^\perp$ and G are the corresponding eigenspaces.

2. (a)

$$(\mathcal{U}f, \mathcal{U}g) = (f, g) \quad \forall\, f, g \in \mathcal{H}$$

$$\Big\downarrow \text{turn over rule}$$

$$(\mathcal{U}^{\dagger}\mathcal{U}f, g) = (\mathbb{1}f, g) \quad \forall\, f, g \in \mathcal{H}.$$

Thus, $$\mathcal{U}^{\dagger}\mathcal{U} = \mathbb{1}.$$

Since the domain and range of $\mathcal{U}$ is the entire Hilbert space, $\mathcal{U}^{-1}$ exists and

$$\mathcal{U}^{-1}\mathcal{U} = \mathbb{1}.$$

Thus, $$\mathcal{U}^{-1} = \mathcal{U}^{\dagger}.$$

(b) $\left(U(\alpha f)-\alpha Uf,\ U(\alpha f)-\alpha Uf\right)$

$\underset{\text{of inner product}}{\overset{\text{bilinearity}}{=}} \left(U(\alpha f), U(\alpha f)\right)-\alpha\left(U(\alpha f), Uf\right)$

$-\alpha^*\left(Uf, U(\alpha f)\right)+\alpha^*\alpha\left(Uf, Uf\right).$

From Eq. (1), we obtain (after dropping Us),

$$\left(U(\alpha f)-\alpha Uf,\ U(\alpha f)-\alpha Uf\right)$$

$$=(\alpha f,\alpha f)-\alpha(\alpha f,f)-\alpha^*(f,\alpha f)$$

$$+\alpha^*\alpha(f,f)=\cancel{\alpha^*\alpha(f,f)}-\cancel{\alpha\alpha^*(f,f)}$$

$$-\cancel{\alpha^*\alpha(f,f)}+\cancel{\alpha^*\alpha(f,f)}=0.$$

Thus, $\left(U(\alpha f)-\alpha Uf,\ U(\alpha f)-\alpha Uf\right)=0.$

In general, the condition $(g,g)=0$ implies that $g=\Theta$ (zero vector). Thus,

$$U(\alpha f)-\alpha Uf=\Theta \quad \text{or}$$

$$\underline{U(\alpha f)=\alpha Uf},\ \text{which is Eq. (2).}$$

(c) $\mathcal{U} f_1 = \lambda_1 f_1,$

$\mathcal{U} f_2 = \lambda_2 f_2, \quad \lambda_2 \neq \lambda_1.$

$$(\mathcal{U} f_1, \mathcal{U} f_2) = (\lambda_1 f_1, \lambda_2 f_2)$$
$$= \lambda_1^* \lambda_2 (f_1, f_2).$$

On the other hand,

$$(\mathcal{U} f_1, \mathcal{U} f_2) = (f_1, f_2).$$

We obtain,

$$\lambda_1^* \lambda_2 (f_1, f_2) = (f_1, f_2).$$

In Assignment 4, we proved that

(Problem 3(b)) $|\lambda_1| = |\lambda_2| = 1$, so that $\lambda_1 = e^{i\alpha_1}$, $\lambda_2 = e^{i\alpha_2}$, for some $0 \leq \alpha_1, \alpha_2 < 2\pi$.

Suppose $(f_1, f_2) \neq 0$. In this case,

$$\lambda_1^* \lambda_2 = e^{i(\alpha_2-\alpha_1)} = 1.$$

Since both α_1 and α_2 belong to the interval $[0, 2\pi)$,

$e^{i(\alpha_2-\alpha_1)} = 1$ implies that

$\alpha_2 - \alpha_1 = 0$ or that $\alpha_1 = \alpha_2$.

Thus,

$\lambda_1 = \lambda_2$, contrary to our assumption that $\lambda_1 \neq \lambda_2$.

We cannot assume that $(f_1, f_2) \neq 0$.

In consequence,

$(f_1, f_2) = 0$ when $\lambda_1 \neq \lambda_2$.

3. (a) Consider an eigenvalue problem

$$Af = \lambda f \quad (f\text{ - eigenvector, } \lambda\text{ - eigenvalue}).$$

$$(f, Af) = (f, \lambda f) = \lambda (f, f) = \lambda \|f\|^2$$

$$\|$$

$$(A^\dagger f, f) = (-Af, f) = -(\lambda f, f) = $$
$$= -\lambda^* (f, f) = -\lambda^* \|f\|^2.$$

Thus, $\lambda \|f\|^2 = -\lambda^* \|f\|^2$. Since $f \neq \Theta$ and $\|f\| > 0$, we have

$$\lambda = -\lambda^*.$$

If $\lambda = a + bi$, where $a = \mathrm{Re}\,\lambda$, $b = \mathrm{Im}\,\lambda$, we can write

$$a + bi = -(a - bi) = -a + bi.$$

Thus,

$$a = -a \quad \text{or} \quad 2a = 0, \text{ i.e., } a = 0.$$

$\lambda = bi \rightarrow$ belongs to the imaginary axis.

(b) Consider

$$Af_1 = \lambda_1 f_1, \quad Af_2 = \lambda_2 f_2,$$
$$\lambda_1 \neq \lambda_2 .$$

We obtain

$$(f_1, Af_2) = (f_1, \lambda_2 f_2) = \lambda_2 (f_1, f_2)$$

$$(A^\dagger f_1, f_2) = (-Af_1, f_2) = -\lambda_1^* (f_1, f_2)$$

We already know from (a) that λ_1 and λ_2 belong to the imaginary axis.

In particular, $\lambda_1 = -\lambda_1^*$.

Thus,

$$\lambda_2 (f_1, f_2) = -\lambda_1^* (f_1, f_2) = \lambda_1 (f_1, f_2),$$

or

$$(\lambda_1 - \lambda_2)(f_1, f_2) = 0 \text{ , so that}$$

either $\lambda_1 - \lambda_2 = 0$ or $(f_1, f_2) = 0$.

The former is not possible, since $\lambda_1 \neq \lambda_2$.

Thus,

$$(f_1, f_2) = 0.$$

(c) We must prove that

$$\left(\frac{\partial}{\partial q}\right)^{+} = -\frac{\partial}{\partial q}.$$

In general, $\hat{F}^{+}$ is defined by

$$\int_{\Gamma} f^* \hat{F} g \, d\tau = \int_{\Gamma} (\hat{F}^{+} f)^* g \, d\tau.$$

In our case, $\hat{F} = \frac{\partial}{\partial\varphi}$, $\Gamma = [0, 2\pi]$.

$$\int_0^{2\pi} f^* \frac{\partial g}{\partial\varphi} d\varphi \underset{\text{by parts}}{\overset{\text{integrating}}{=\!=}} f^* g \Big|_{\varphi=0}^{\varphi=2\pi}$$

$$- \int_0^{2\pi} \left(\frac{\partial f}{\partial\varphi}\right)^* g \, d\varphi .$$

$f(0) = f(2\pi)$, $g(0) = g(2\pi)$. Thus,

$$\int_0^{2\pi} f^* \frac{\partial g}{\partial\varphi} d\varphi = - \int_0^{2\pi} \left(\frac{\partial f}{\partial\varphi}\right)^* g \, d\varphi$$

$$= \int_0^{2\pi} \left(-\frac{\partial f}{\partial\varphi}\right)^* g \, d\varphi \equiv \int_0^{2\pi} \left[\left(\frac{\partial}{\partial\varphi}\right)^\dagger f\right]^* g \, d\varphi .$$

Thus,

$$\left(\frac{\partial}{\partial\varphi}\right)^\dagger = -\frac{\partial}{\partial\varphi} .$$

$\therefore$

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