CEM 991 **Fall 2021**

Assignment No. 2 (25 pts.) **Due Date: October 1, 2021 (Friday)**

1. (3 pts.) Let q_ℓ and p_ℓ, $\ell = 1, \ldots, f$, designate the generalized coordinates and conjugate momenta, respectively, and let $\{\ ,\ \}$ be the Poisson bracket. Prove the following relationships:

   (a) $\{q_\ell, q_{\ell'}\} = 0$,

   (b) $\{p_\ell, p_{\ell'}\} = 0$,

   (c) $\{q_\ell, p_{\ell'}\} = \delta_{\ell\ell'}$.

2. (5 pts.) Let F_i, G_i, $i = 1, 2$, and G be functions of $q_1, \ldots, q_f, p_1, \ldots, p_f$, and t and let $\{\ ,\ \}$ designate the Poisson bracket. Prove the following properties of the Poisson bracket:

   (a) $\{F_1, F_2\} = -\{F_2, F_1\}$,

   (b) $\{F_1F_2, G\} = F_1\{F_2, G\} + \{F_1, G\}F_2$,

   (c) $\{F_1F_2, G_1G_2\} = F_1G_1\{F_2, G_2\} + F_1\{F_2, G_1\}G_2 + G_1\{F_1, G_2\}F_2 + \{F_1, G_1\}G_2F_2$.

   Hint: To prove (c), use (a) and (b) replacing G by G_1G_2.

3. (5 pts.) It can be shown that the Poisson brackets have the following two properties:

   (a) $\{\{F_1, F_2\}, F_3\} + \{\{F_2, F_3\}, F_1\} + \{\{F_3, F_1\}, F_2\} = 0$ (Jacobi's identity),

   (b) $\dfrac{\partial}{\partial t}\{F_1, F_2\} = \left\{\dfrac{\partial}{\partial t}F_1, F_2\right\} + \left\{F_1, \dfrac{\partial}{\partial t}F_2\right\}$,

   where F_i, $i = 1 - 3$, are functions of $q = (q_1, \ldots, q_f)$, $p = (p_1, \ldots, p_f)$, and t. Prove property (b). Use properties (a) and (b) to prove the following theorem (known as the Poisson-Jacobi theorem): *If two dynamical variables F_1 and F_2 are constants of motion, i.e., $F_1(q, p, t) = const$ and $F_2(q, p, t) = const$, then the Poisson bracket of F_1 and F_2, i.e., $\{F_1, F_2\}$, is also a constant of motion.*
   Hint: Use the equation of motion in terms of the Poisson brackets for the dynamical variable $F = \{F_1, F_2\}$ and use property (a) (Jacobi's identity) with $F_3 = H$, where H is the Hamilton function. Next, use the equation of motion in terms of the Poisson brackets for dynamical variables F_1 and F_2, knowing that $dF_1/dt = 0$ and $dF_2/dt = 0$. Finally, use property (b) to demonstrate that $(d/dt)\{F_1, F_2\} = 0$.

4. (5 pts.) Use the results of Problems 1 and 2 to demonstrate that

$$\{L_x, L_y\} = L_z, \tag{1}$$

$$\{L_y, L_z\} = L_x, \tag{2}$$

$$\{L_z, L_x\} = L_y, \tag{3}$$

where L_x, L_y, and L_z are the x, y, and z components of the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$[#)], respectively, and $\{\ ,\ \}$ is the Poisson bracket. It is enough if you derive Eq. (1), since the remaining two equations, Eqs. (2) and (3), can be obtained by cyclic permutations of variables x, y, and z. Use the above Eqs. (1)–(3) and the Poisson-Jacobi theorem formulated in Problem 3 to show that if any two components of the angular momentum are constant during motion, then the third one must be constant as well.

5. (7 pts.) Prove that relationships (a)–(c) listed in Problem 2 remain valid if we replace F_i, G_i, $i = 1, 2$, and G by linear operators and the Poisson brackets $\{\ ,\ \}$ by commutators $[\ ,\ ]$. Prove that Jacobi's identity (relationship (a) in Problem 3) remains valid if we replace F_i, $i = 1 - 3$, by linear operators and the Poisson brackets $\{\ ,\ \}$ by commutators $[\ ,\ ]$.

[#)] Bold symbols designate vector quantities.

SOLUTIONS

1. The Poisson bracket of two dynamical variables $F = F(q,p,t)$ and $G = G(q,p,t)$ is defined as follows,

$$\{F,G\} = \sum_{k=1}^{f}\left(\frac{\partial F}{\partial q_k}\frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial G}{\partial q_k}\right).$$

(a) $$\{q_l, q_{l'}\} = \sum_{k=1}^{f}\left(\frac{\partial q_l}{\partial q_k}\frac{\partial q_{l'}}{\partial p_k} - \frac{\partial q_l}{\partial p_k}\frac{\partial q_{l'}}{\partial q_k}\right)$$

$$= 0, \text{ since } \frac{\partial q_{l'}}{\partial p_k} = \frac{\partial q_l}{\partial p_k} = 0.$$

(b) $$\{p_l, p_{l'}\} = \sum_{k=1}^{f}\left(\frac{\partial p_l}{\partial q_k}\frac{\partial p_{l'}}{\partial p_k} - \frac{\partial p_l}{\partial p_k}\frac{\partial p_{l'}}{\partial q_k}\right)$$

$$= 0, \text{ since } \frac{\partial p_l}{\partial q_k} = \frac{\partial p_{l'}}{\partial q_k} = 0.$$

(c) $$\{q_l, p_{l'}\} = \sum_{k=1}^{f}\left(\frac{\partial q_l}{\partial q_k}\frac{\partial p_{l'}}{\partial p_k} - \overbrace{\frac{\partial q_l}{\partial p_k}}^{0}\overbrace{\frac{\partial p_{l'}}{\partial q_k}}^{0}\right)$$

$$= \sum_{k=1}^{f} \delta_{kl}\delta_{kl'},$$

where $\delta_{kl} = \begin{cases} 0 \text{ for } k \neq l \\ 1 \text{ for } k = l \end{cases}$.

When $l = l'$, we obtain

$$\{q_l, p_l\} = \sum_{k=1}^{f} (\delta_{kl})^2 = (\delta_{ll})^2 = 1,$$

When $l \neq l'$,

$$\{q_l, p_{l'}\} = \underset{(k=l)}{\delta_{ll}} \overset{0}{\overset{\parallel}{\delta_{ll'}}} + \overset{0}{\overset{\parallel}{\delta_{l'l}}} \underset{(k=l')}{\delta_{l'l'}}$$

$$= 0.$$

In consequence, $\{q_l, p_{l'}\} = \delta_{ll'}$.

$\therefore$

2. (a) $\{F_2, F_1\} = \sum_{k=1}^{f} \left(\frac{\partial F_2}{\partial q_k} \frac{\partial F_1}{\partial p_k} - \frac{\partial F_2}{\partial p_k} \frac{\partial F_1}{\partial q_k} \right)$

$$= - \sum_{k=1}^{f} \left(\frac{\partial F_1}{\partial q_k} \frac{\partial F_2}{\partial p_k} - \frac{\partial F_1}{\partial p_k} \frac{\partial F_2}{\partial q_k} \right)$$

$$= - \{F_1, F_2\}.$$

(b) $$\{F_1F_2, G\} = \sum_{k=1}^{f}\left(\frac{\partial(F_1F_2)}{\partial q_k}\frac{\partial G}{\partial p_k} - \frac{\partial(F_1F_2)}{\partial p_k}\frac{\partial G}{\partial q_k}\right) = \sum_{k=1}^{f}\left(\underline{\frac{\partial F_1}{\partial q_k}F_2\frac{\partial G}{\partial p_k}} + \underline{\underline{F_1\frac{\partial F_2}{\partial q_k}\frac{\partial G}{\partial p_k}}} - \underline{\frac{\partial F_1}{\partial p_k}F_2\frac{\partial G}{\partial q_k}} - \underline{\underline{F_1\frac{\partial F_2}{\partial p_k}\frac{\partial G}{\partial q_k}}}\right) =$$

$$= \sum_{k=1}^{f}\left[\left(\frac{\partial F_1}{\partial q_k}\frac{\partial G}{\partial p_k} - \frac{\partial F_1}{\partial p_k}\frac{\partial G}{\partial q_k}\right)F_2 + F_1\left(\frac{\partial F_2}{\partial q_k}\frac{\partial G}{\partial p_k} - \frac{\partial F_2}{\partial p_k}\frac{\partial G}{\partial q_k}\right)\right]$$

$$= \left[\sum_{k=1}^{f}\left(\frac{\partial F_1}{\partial q_k}\frac{\partial G}{\partial p_k} - \frac{\partial F_1}{\partial p_k}\frac{\partial G}{\partial q_k}\right)\right]F_2 + F_1\sum_{k=1}^{f}\left(\frac{\partial F_2}{\partial q_k}\frac{\partial G}{\partial p_k} - \frac{\partial F_2}{\partial p_k}\frac{\partial G}{\partial q_k}\right)$$

$$= \{F_1, G\}F_2 + F_1\{F_2, G\}.$$

(c) $\{F_1F_2, G_1G_2\} \overset{(b)}{\underset{G=G_1G_2}{=\!=}} F_1\{F_2, G_1G_2\}$

$+ \{F_1, G_1G_2\}F_2 \overset{(a)}{=\!=} -F_1\{G_1G_2, F_2\}$

$- \{G_1G_2, F_1\}F_2 \overset{(b)}{=\!=}$

$\overset{(b)}{=\!=} -F_1\left(G_1\{G_2, F_2\} + \{G_1, F_2\}G_2\right)$

$- \left(G_1\{G_2, F_1\} + \{G_1, F_1\}G_2\right)F_2$

$= -F_1G_1\{G_2, F_2\} - F_1\{G_1, F_2\}G_2$

$- G_1\{G_2, F_1\}F_2 - \{G_1, F_1\}G_2F_2$

$\overset{(a)}{=} F_1G_1\{F_2, G_2\} + F_1\{F_2, G_1\}G_2$

$+ G_1\{F_1, G_2\}F_2 + \{F_1, G_1\}G_2F_2$

$\therefore$

3. Proof of (b):

$$\frac{\partial}{\partial t}\{F_1, F_2\} = \frac{\partial}{\partial t}\sum_{k=1}^{f}\left(\frac{\partial F_1}{\partial q_k}\frac{\partial F_2}{\partial p_k} - \frac{\partial F_1}{\partial p_k}\frac{\partial F_2}{\partial q_k}\right)$$

$$= \sum_{k=1}^{f}\left[\frac{\partial}{\partial t}\left(\frac{\partial F_1}{\partial q_k}\frac{\partial F_2}{\partial p_k}\right) - \frac{\partial}{\partial t}\left(\frac{\partial F_1}{\partial p_k}\frac{\partial F_2}{\partial q_k}\right)\right]$$

$$= \sum_{k=1}^{f}\left[\frac{\partial}{\partial t}\left(\frac{\partial F_1}{\partial q_k}\right)\frac{\partial F_2}{\partial p_k} + \frac{\partial F_1}{\partial q_k}\frac{\partial}{\partial t}\left(\frac{\partial F_2}{\partial p_k}\right)\right.$$

$$\left. - \frac{\partial}{\partial t}\left(\frac{\partial F_1}{\partial p_k}\right)\frac{\partial F_2}{\partial q_k} - \frac{\partial F_1}{\partial p_k}\frac{\partial}{\partial t}\left(\frac{\partial F_2}{\partial q_k}\right)\right]$$

$$= \sum_{k=1}^{f}\left[\overset{(1)}{\frac{\partial}{\partial q_k}\left(\frac{\partial F_1}{\partial t}\right)\frac{\partial F_2}{\partial p_k}} + \overset{(2)}{\frac{\partial F_1}{\partial q_k}\frac{\partial}{\partial p_k}\left(\frac{\partial F_2}{\partial t}\right)}\right.$$

$$\left. - \overset{(3)}{\frac{\partial}{\partial p_k}\left(\frac{\partial F_1}{\partial t}\right)\frac{\partial F_2}{\partial q_k}} - \overset{(4)}{\frac{\partial F_1}{\partial p_k}\frac{\partial}{\partial q_k}\left(\frac{\partial F_2}{\partial t}\right)}\right]$$

(reordering terms and splitting $\sum_k$)

$$= \sum_{k=1}^{f} \left[\frac{\partial}{\partial q_k}\left(\frac{\partial F_1}{\partial t}\right) \frac{\partial F_2}{\partial p_k} - \frac{\partial}{\partial p_k}\left(\frac{\partial F_1}{\partial t}\right) \frac{\partial F_2}{\partial q_k} \right]$$

$$+ \sum_{k=1}^{f} \left[\frac{\partial F_1}{\partial q_k} \frac{\partial}{\partial p_k}\left(\frac{\partial F_2}{\partial t}\right) - \frac{\partial F_1}{\partial p_k} \frac{\partial}{\partial q_k}\left(\frac{\partial F_2}{\partial t}\right) \right]$$

$$= \left\{ \frac{\partial F_1}{\partial t}, F_2 \right\} + \left\{ F_1, \frac{\partial F_2}{\partial t} \right\}$$

Proof of the Poisson-Jacobi theorem:

Let us define the dynamical variable

$$F = \{F_1, F_2\} .$$

We know that

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \{F, H\} ,$$

where H is the Hamilton function.

We obtain,

$$\frac{dF}{dt} = \frac{\partial}{\partial t}\{F_1, F_2\} + \{\{F_1, F_2\}, H\}, \quad (A)$$

where

$$\{\{F_1, F_2\}, H\} \overset{(a)}{=\!=} -\{\{F_2, H\}, F_1\}$$

$$-\{\{H, F_1\}, F_2\} = -\{\{F_2, H\}, F_1\}$$

$$+\{\{F_1, H\}, F_2\}. \quad (\mathbf{B})$$

We know that F_1 and F_2 are constants of motion, i.e.

$$\frac{dF_1}{dt} = \frac{\partial F_1}{\partial t} + \{F_1, H\} = 0,$$

$$\frac{dF_2}{dt} = \frac{\partial F_2}{\partial t} + \{F_2, H\} = 0.$$

Thus,

$$\{F_1, H\} = -\frac{\partial F_1}{\partial t}, \quad \{F_2, H\} = -\frac{\partial F_2}{\partial t}.$$

If we insert the above equations into (**B**), we obtain

$$\{\{F_1, F_2\}, H\} = +\left\{\frac{\partial F_2}{\partial t}, F_1\right\}$$

$$-\left\{\frac{\partial F_1}{\partial t}, F_2\right\} = -\left\{F_1, \frac{\partial F_2}{\partial t}\right\}$$

$$-\left\{\frac{\partial F_1}{\partial t}, F_2\right\}. \qquad (C)$$

By combining (A) and (C), we obtain

$$\frac{dF}{dt} = \frac{\partial}{\partial t}\{F_1, F_2\} - \left\{\frac{\partial F_1}{\partial t}, F_2\right\}$$

$$-\left\{F_1, \frac{\partial F_2}{\partial t}\right\} \overset{(b)}{=\!=} 0$$

(we used property (b) as well). Thus, $F = \{F_1, F_2\}$ is a constant of motion.

4. Recall that

$$L_x = yp_z - zp_y,$$
$$L_y = zp_x - xp_z,$$
$$L_z = xp_y - yp_x.$$

From Problem 1, we obtain

$$\{x, x\} = \{x, y\} = \{x, z\} = \ldots = \{z, z\} = 0,$$
$$\{p_x, p_x\} = \{p_x, p_y\} = \{p_x, p_z\} = \ldots = \{p_z, p_z\} = 0,$$
$$\{x, p_x\} = \{y, p_y\} = \{z, p_z\} = 1 \text{ and}$$
$$\{x_j, p_{x_k}\} = 0 \text{ for } j \neq k \; (x_1 = x,$$
$$x_2 = y, \; x_3 = z).$$

We obtain,

$$\{L_x, L_y\} = \{yp_z - zp_y, zp_x - xp_z\}$$
$$= \{yp_z, zp_x\} - \{yp_z, xp_z\}$$
$$- \{zp_y, zp_x\} + \{zp_y, xp_z\},$$

since, in general,

$$\{F_1+F_2, G_1+G_2\} = \{F_1, G_1\} + \{F_1, G_2\} + \{F_2, G_1\} + \{F_2, G_2\}.$$

Now, using the property (c) listed in Problem 2, we find

$$\{yp_z, zp_x\} = yz\{p_z, p_x\} + y\{p_z, z\}p_x + z\{y, p_x\}p_z + \{y, z\}p_x p_z$$

$$= y\{p_z, z\}p_x \text{, since}$$

$\{p_z, p_x\}$, $\{y, p_x\}$, and $\{y, z\}$ vanish.

Thus, $\{yp_z, zp_x\} = -y\{z, p_z\}p_x$

$$= -yp_x.$$

Similar arguments allow us to prove that

$$\{yp_z, xp_z\} = yx\{p_z, p_z\} + y\{p_z, x\}p_z + x\{y, p_z\}p_z + \{y, x\}p_z^2$$

$$= 0,$$

$$\{zp_y, zp_x\} = z^2\{p_y, p_x\} + z\{p_y, z\}p_x$$
$$+ z\{z, p_x\}p_y + \{z, z\}p_x p_y = 0,$$
$$\{zp_y, xp_z\} = zx\{p_y, p_z\} + z\{p_y, x\}p_z$$
$$+ x\{z, p_z\}p_y + \{z, x\}p_z p_y$$
$$= x\{z, p_z\}p_y = xp_y.$$

In consequence,

$$\{L_x, L_y\} = -yp_x + 0 + 0 + xp_y$$
$$= xp_y - yp_x = L_z$$

Notice that the entire calculation has a purely algebraic nature.

We have to prove now that if two components of $\vec{L}$ are constant, then the third one is a constant of motion as well.

Let us assume, for example, that L_x and L_y are constants of motion.

According to the Poisson-Jacobi theorem (Problem 3),

$$\{L_x, L_y\}$$

is also a constant of motion.

But $\{L_x, L_y\} \overset{Eq.(1)}{=} L_z$. Thus, L_z is a constant of motion as well.

$\therefore$

5. Let us use a notation, in which $\hat{F}_i$, $\hat{G}_i$, $i=1,2$, and $\hat{G}$ are the operators representing F_i, G_i, $i=1,2$, and G, respectively.

(a) $[\hat{F}_2,\hat{F}_1] = \hat{F}_2\hat{F}_1 - \hat{F}_1\hat{F}_2 =$

$$= -(\hat{F}_1\hat{F}_2 - \hat{F}_2\hat{F}_1) = -[\hat{F}_1,\hat{F}_2].$$

(b) $[\hat{F}_1\hat{F}_2,\hat{G}] = \hat{F}_1\hat{F}_2\hat{G} - \hat{G}\hat{F}_1\hat{F}_2$

$$\underset{\hat{F}_1\hat{G}\hat{F}_2}{\overset{\text{add and subtract}}{=}} \hat{F}_1\hat{F}_2\hat{G} - \hat{F}_1\hat{G}\hat{F}_2 + \hat{F}_1\hat{G}\hat{F}_2 - \hat{G}\hat{F}_1\hat{F}_2 =$$

$$= \hat{F}_1(\hat{F}_2\hat{G} - \hat{G}\hat{F}_2) + (\hat{F}_1\hat{G} - \hat{G}\hat{F}_1)\hat{F}_2$$

$$= \hat{F}_1[\hat{F}_2,\hat{G}] + [\hat{F}_1,\hat{G}]\hat{F}_2 .$$

(c) $$[\hat{F}_1\hat{F}_2, \hat{G}_1\hat{G}_2] \underset{\hat{G}=\hat{G}_1\hat{G}_2}{\overset{(b)}{=}} \hat{F}_1[\hat{F}_2, \hat{G}_1\hat{G}_2]$$

$$+ [\hat{F}_1, \hat{G}_1\hat{G}_2]\hat{F}_2 \overset{(a)}{=} -\hat{F}_1[\hat{G}_1\hat{G}_2, \hat{F}_2]$$

$$- [\hat{G}_1\hat{G}_2, \hat{F}_1]\hat{F}_2 \overset{(b)}{=}$$

$$\overset{(b)}{=} -\hat{F}_1\left(\hat{G}_1[\hat{G}_2, \hat{F}_2] + [\hat{G}_1, \hat{F}_2]\hat{G}_2\right)$$

$$- \left(\hat{G}_1[\hat{G}_2, \hat{F}_1] + [\hat{G}_1, \hat{F}_1]\hat{G}_2\right)\hat{F}_2$$

$$\overset{(a)}{=} \hat{F}_1\hat{G}_1[\hat{F}_2, \hat{G}_2] + \hat{F}_1[\hat{F}_2, \hat{G}_1]\hat{G}_2$$

$$+ \hat{G}_1[\hat{F}_1, \hat{G}_2]\hat{F}_2 + [\hat{F}_1, \hat{G}_1]\hat{G}_2\hat{F}_2 .$$

Verification:

$$\hat{F}_1\hat{G}_1[\hat{F}_2, \hat{G}_2] + \hat{F}_1[\hat{F}_2, \hat{G}_1]\hat{G}_2$$
$$+ \hat{G}_1[\hat{F}_1, \hat{G}_2]\hat{F}_2 + [\hat{F}_1, \hat{G}_1]\hat{G}_2\hat{F}_2 =$$

$$= \hat{F}_1\hat{G}_1(\hat{F}_2\hat{G}_2-\hat{G}_2\hat{F}_2)+\hat{F}_1(\hat{F}_2\hat{G}_1-\hat{G}_1\hat{F}_2)\hat{G}_2$$

$$+\hat{G}_1(\hat{F}_1\hat{G}_2-\hat{G}_2\hat{F}_1)\hat{F}_2+(\hat{F}_1\hat{G}_1-\hat{G}_1\hat{F}_1)\hat{G}_2\hat{F}_2$$

$$= \underline{\cancel{\hat{F}_1\hat{G}_1\hat{F}_2\hat{G}_2}}-\underline{\underline{\cancel{\hat{F}_1\hat{G}_1\hat{G}_2\hat{F}_2}}}+\hat{F}_1\hat{F}_2\hat{G}_1\hat{G}_2$$

$$-\underline{\cancel{\hat{F}_1\hat{G}_1\hat{F}_2\hat{G}_2}}+\cancel{\hat{G}_1\hat{F}_1\hat{G}_2\hat{F}_2}-\hat{G}_1\hat{G}_2\hat{F}_1\hat{F}_2$$

$$+\underline{\underline{\cancel{\hat{F}_1\hat{G}_1\hat{G}_2\hat{F}_2}}}-\cancel{\hat{G}_1\hat{F}_1\hat{G}_2\hat{F}_2}$$

$$= \hat{F}_1\hat{F}_2\hat{G}_1\hat{G}_2-\hat{G}_1\hat{G}_2\hat{F}_1\hat{F}_2=[\hat{F}_1\hat{F}_2,\hat{G}_1\hat{G}_2] \quad \therefore$$

<u>Jacobi's identity</u>:

We have to show that

$$[[F_1,F_2],F_3]+[[F_2,F_3],F_1]+$$
$$+[[F_3,F_1],F_2]=0.$$

We obtain

$$[F_1F_2 - F_2F_1, F_3] + [F_2F_3 - F_3F_2, F_1] + [F_3F_1 - F_1F_3, F_2] =$$

$$= [F_1F_2, F_3] - [F_2F_1, F_3] + [F_2F_3, F_1] - [F_3F_2, F_1] + [F_3F_1, F_2] - [F_1F_3, F_2]$$

$$= \cancel{F_1F_2F_3} - \cancel{F_3F_1F_2} - \cancel{F_2F_1F_3}$$

$$+ \cancel{F_3F_2F_1} + \cancel{F_2F_3F_1} - \cancel{F_1F_2F_3}$$

$$- \cancel{F_3F_2F_1} + \cancel{F_1F_3F_2} + \cancel{F_3F_1F_2}$$

$$- \cancel{F_2F_3F_1} - \cancel{F_1F_3F_2} + \cancel{F_2F_1F_3} = 0$$