

1. (4 pts.) Consider the classical particle with mass m . Let x , y , and z define the Cartesian components of the radius vector[#] $\mathbf{r}$ from the origin to the particle and let r , θ , and ϕ designate the spherical components of $\mathbf{r}$, so that

$$x = r \sin \theta \cos \phi, \quad (1)$$

$$y = r \sin \theta \sin \phi, \quad (2)$$

$$z = r \cos \theta. \quad (3)$$

Derive the formula for the kinetic energy of the particle m ,

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2). \quad (4)$$

Recall that the dot above a symbol designates the time derivative $\frac{d}{dt}$.

2. (8 pts.) Consider the classical particle with mass m under the influence of a force derived from the potential $V(\mathbf{r}, t)$ (no constraints). You have learned in class that the canonical momenta p_ℓ , $\ell = 1, 2, 3$, are related to Cartesian components of the linear momentum $\mathbf{p} = m\dot{\mathbf{r}}$ as follows:

$$p_\ell = p_x \frac{\partial x}{\partial q_\ell} + p_y \frac{\partial y}{\partial q_\ell} + p_z \frac{\partial z}{\partial q_\ell}. \quad (5)$$

Let r , θ , and ϕ designate the spherical components of $\mathbf{r}$, so that $q_1 = r$, $q_2 = \theta$, and $q_3 = \phi$.

- (a) Using the general definition of the canonical momentum p_ℓ ,

$$p_\ell = \frac{\partial L}{\partial \dot{q}_\ell}, \quad L = T - V, \quad (6)$$

and the fact that the kinetic energy T expressed in terms of r , θ , and ϕ is given by Eq. (4) (see Problem 1), prove that

$$p_r = m\dot{r}, \quad (7)$$

$$p_\theta = mr^2\dot{\theta}, \quad (8)$$

$$p_\phi = mr^2\sin^2\theta\dot{\phi}. \quad (9)$$

- (b) Show that

$$p_r = \mathbf{p} \cdot \frac{\mathbf{r}}{r}, \quad (10)$$

so that p_r has an interpretation of a linear momentum along the radial direction.

[#]) Bold symbols designate vector quantities.

- (c) Show that p_ϕ can be interpreted as the z -component of the angular momentum vector

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}. \quad (11)$$

In other words, L_z is conjugate to ϕ . *Hint:* Note that

$$L_z = xp_y - yp_x = m(x\dot{y} - y\dot{x}), \quad (12)$$

and use the relationships (1)–(3) and the chain rule to express $\dot{x}$, $\dot{y}$, and $\dot{z}$ in terms of $\dot{r}$, $\dot{\theta}$, and $\dot{\phi}$.

- (d) By using the equations presented in (a) and (b) and the definition of the Hamiltonian,

$$H = \sum_{\ell=1}^3 p_\ell v_\ell - L, \quad (13)$$

where $L = T - V$ is the Lagrangian and $v_\ell = \dot{q}_\ell$ (recall that v_ℓ has to be expressed in terms of q_ℓ and p_ℓ , $\ell = 1, 2, 3$), show that

$$H = \frac{1}{2m} \left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2 \theta} \right) + V. \quad (14)$$

Show that the same expression for H is obtained by using the formula

$$H = T + V. \quad (15)$$

Explain why Eqs. (13) and (15) lead to the same result.

3. (8 pts.) The force that governs the motion of a harmonic oscillator with mass m and the angular frequency ω is defined as

$$F_x = -kx, \quad (16)$$

where $k = m\omega^2$.

- (a) Calculate the potential function V [assume that $V(x = 0) = 0$].
 (b) Calculate the Lagrangian and the Hamiltonian.
 (c) Show that the Lagrange equation reduces in this case to ordinary Newton's equation

$$m\ddot{x} = F_x, \quad (17)$$

where F_x is defined by Eq. (16).

- (d) Solve Newton's equation [Eq. (17)], i.e. determine $x = x(t)$, assuming the following initial conditions: $x(t = 0) = 0$, $\dot{x}(t = 0) = v_0$. Use the solution $x(t)$, the definition of the kinetic energy T , and the formula for the potential function V [see item (a) above] to show that

$$T + V = \text{const.} \quad (18)$$

What is the value of $T + V$ in this case?

4. (5 pts.) The force on a particle with charge q is given by the well-known Lorentz formula

$$\mathbf{F} = q \left[\mathbf{E} + \frac{1}{c}(\mathbf{v} \times \mathbf{B}) \right], \quad (19)$$

where

$$\mathbf{E} = -\nabla\phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \quad (20)$$

and

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (21)$$

are, respectively, the electric and magnetic field vectors, ϕ is the electric potential, $\mathbf{A}$ is the magnetic vector potential, and $\mathbf{v} = (\dot{x}, \dot{y}, \dot{z})$ is the velocity. The velocity-dependent potential U corresponding to Lorentz force $\mathbf{F}$, Eq. (19), is given by the formula,

$$U = q\phi - \frac{q}{c} \mathbf{A} \cdot \mathbf{v}. \quad (22)$$

Show that the following equation,

$$F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt} \frac{\partial U}{\partial \dot{x}}, \quad (23)$$

is true.

Hint: Calculate $\frac{\partial U}{\partial x}$ and $\frac{\partial U}{\partial \dot{x}}$ using formula (22) and compare the result with the formula for the x component of $\mathbf{F}$ resulting from Eqs. (19)–(21). Recall that for any vector field $\mathbf{G} = \mathbf{G}(x, y, z, t)$,

$$(\nabla \times \mathbf{G})_x = \frac{\partial G_z}{\partial y} - \frac{\partial G_y}{\partial z}, \quad (24)$$

$$(\nabla \times \mathbf{G})_y = \frac{\partial G_x}{\partial z} - \frac{\partial G_z}{\partial x}, \quad (25)$$

$$(\nabla \times \mathbf{G})_z = \frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y}. \quad (26)$$

SOLUTIONS

1. The kinetic energy is defined as follows,

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

From Eqs. (1) - (3) it immediately follows that

$$\begin{aligned} \dot{x} = & \dot{r} \sin \theta \cos \phi + r \cos \theta \dot{\theta} \cos \phi \\ & - r \sin \theta \sin \phi \dot{\phi}, \end{aligned}$$

$$\begin{aligned} \dot{y} = & \dot{r} \sin \theta \sin \phi + r \cos \theta \dot{\theta} \sin \phi \\ & + r \sin \theta \cos \phi \dot{\phi}, \end{aligned}$$

$$\dot{z} = \dot{r} \cos \theta - r \sin \theta \dot{\theta}$$

Thus, we obtain

$$\begin{aligned} \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = & \dot{r}^2 \sin^2 \theta \cos^2 \phi \\ & + \underline{\underline{r^2 \cos^2 \theta \cos^2 \phi \dot{\theta}^2}} + \underline{\underline{r^2 \sin^2 \theta \sin^2 \phi \dot{\phi}^2}} \end{aligned}$$

$$\begin{aligned}
 & + \cancel{2 r \dot{r} \dot{\theta} \sin \theta \cos \theta \cos^2 \varphi} \\
 & - \cancel{2 r \dot{r} \dot{\varphi} \sin^2 \theta \sin \varphi \cos \varphi} \\
 & - \cancel{2 r^2 \sin \theta \cos \theta \sin \varphi \cos \varphi \dot{\theta} \dot{\varphi}} \\
 & + \underline{\dot{r}^2 \sin^2 \theta \sin^2 \varphi} + \underline{\underline{r^2 \cos^2 \theta \sin^2 \varphi \dot{\theta}^2}} \\
 & + \underline{\underline{r^2 \sin^2 \theta \cos^2 \varphi \dot{\varphi}^2}} \\
 & + \cancel{2 r \dot{r} \dot{\theta} \sin \theta \cos \theta \sin^2 \varphi} \\
 & + \cancel{2 r \dot{r} \dot{\varphi} \sin^2 \theta \sin \varphi \cos \varphi} \\
 & + \cancel{2 r^2 \sin \theta \cos \theta \sin \varphi \cos \varphi \dot{\theta} \dot{\varphi}} \\
 & + \dot{r}^2 \cos^2 \theta + r^2 \sin^2 \theta \dot{\theta}^2 \\
 & - 2 r \dot{r} \dot{\theta} \sin \theta \cos \theta \quad (1.1)
 \end{aligned}$$

By using the well-known formula

$$\sin^2 \alpha + \cos^2 \alpha = 1, \quad (1.2)$$

We can simplify Eq. (1.1) as follows,

$$\begin{aligned}\dot{x}^2 + \dot{y}^2 + \dot{z}^2 &= \dot{r}^2 \sin^2 \Theta + \underline{\underline{r^2 \dot{\Theta}^2 \cos^2 \Theta}} \\ &+ r^2 \dot{\varphi}^2 \sin^2 \Theta + \underline{\underline{2r\dot{r}\dot{\Theta} \sin \Theta \cos \Theta}} \\ &+ \underline{\underline{\dot{r}^2 \cos^2 \Theta}} + \underline{\underline{r^2 \sin^2 \Theta \dot{\Theta}^2}} \\ &- \underline{\underline{2r\dot{r}\dot{\Theta} \sin \Theta \cos \Theta}}\end{aligned}$$

Eq. (1.2)

$$\dot{r}^2 + r^2 \dot{\Theta}^2 + r^2 \sin^2 \Theta \dot{\varphi}^2$$

As a result,

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\Theta}^2 + r^2 \sin^2 \Theta \dot{\varphi}^2) \quad \therefore$$

2. (a) Since V depends on $\vec{r}$ and t and does not depend on $\dot{\vec{r}}$, we have

$$\frac{\partial V}{\partial \dot{q}_l} = 0$$

In consequence,

$$p_l = \frac{\partial \mathcal{L}}{\partial \dot{q}_l} = \frac{\partial T}{\partial \dot{q}_l}$$

In particular,

$$p_r = \frac{\partial T}{\partial \dot{r}}, \quad p_\theta = \frac{\partial T}{\partial \dot{\theta}}, \quad p_\phi = \frac{\partial T}{\partial \dot{\phi}}.$$

From Eq. (4) (cf. Problem 1), we obtain

$$p_r = \frac{\partial T}{\partial \dot{r}} = \frac{1}{2} m 2 \dot{r} = m \dot{r},$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = \frac{1}{2} m r^2 2 \dot{\theta} = m r^2 \dot{\theta},$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \frac{1}{2} m r^2 \sin^2 \theta 2 \dot{\phi} = m r^2 \sin^2 \theta \dot{\phi}.$$

(b) From Eq. (5), we obtain

$$p_r = p_x \frac{\partial x}{\partial r} + p_y \frac{\partial y}{\partial r} + p_z \frac{\partial z}{\partial r}.$$

Now,

$$\frac{\partial x}{\partial r} = \sin \theta \cos \phi = \frac{x}{r},$$

$$\frac{\partial y}{\partial r} = \sin \theta \sin \phi = \frac{y}{r},$$

$$\frac{\partial z}{\partial r} = \cos \Theta = \frac{z}{r}$$

This means that

$$\begin{aligned} p_r &= p_x \frac{x}{r} + p_y \frac{y}{r} + p_z \frac{z}{r} = \\ &= \vec{p} \cdot \frac{\vec{r}}{r} \end{aligned}$$

(c) The z-component of $\vec{L} = \vec{r} \times \vec{p}$ equals

$$L_z = x p_y - y p_x = m(x \dot{y} - y \dot{x})$$

We have already demonstrated
(see the solution of Problem 1) that

$$\begin{aligned} \dot{x} &= \dot{r} \sin \Theta \cos \varphi + r \cos \Theta \cos \varphi \dot{\Theta} \\ &\quad - r \sin \Theta \sin \varphi \dot{\varphi} \end{aligned}$$

$$\begin{aligned} \dot{y} &= \dot{r} \sin \Theta \sin \varphi + r \cos \Theta \sin \varphi \dot{\Theta} \\ &\quad + r \sin \Theta \cos \varphi \dot{\varphi} \end{aligned}$$

We also know that

$$x = r \sin \theta \cos \varphi,$$

$$y = r \sin \theta \sin \varphi.$$

From the above four equations, we obtain

$$\begin{aligned} x \dot{y} - y \dot{x} &= \cancel{r \dot{r} \sin^2 \theta \sin \varphi \cos \varphi} \\ &+ \cancel{r^2 \sin \theta \cos \theta \sin \varphi \cos \varphi \dot{\theta}} \\ &+ \underline{r^2 \sin^2 \theta \cos^2 \varphi \dot{\varphi}} \\ &- \cancel{r \dot{r} \sin^2 \theta \sin \varphi \cos \varphi} \\ &- \cancel{r^2 \sin \theta \cos \theta \sin \varphi \cos \varphi \dot{\theta}} \\ &+ \underline{r^2 \sin^2 \theta \sin^2 \varphi \dot{\varphi}} \\ &= r^2 \sin^2 \theta (\sin^2 \varphi + \cos^2 \varphi) \dot{\varphi} \\ &= r^2 \sin^2 \theta \dot{\varphi}. \end{aligned}$$

Thus,

$$\begin{aligned} L_z &= m(x\dot{y} - y\dot{x}) = m r^2 \sin^2 \Theta \dot{\varphi} \\ &= p_\varphi \end{aligned}$$

(d) Let us begin with the original definition of H ,

$$H = \sum_{l=1}^3 p_l v_l - L =$$

$$= p_r \dot{r} + p_\Theta \dot{\Theta} + p_\varphi \dot{\varphi} - T + V \quad (2.1)$$

From Eqs. (7) - (9), we obtain,

$$\dot{r} = \frac{p_r}{m}, \quad \dot{\Theta} = \frac{p_\Theta}{m r^2}, \quad \dot{\varphi} = \frac{p_\varphi}{m r^2 \sin^2 \Theta},$$

so that

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$$p_r \dot{r} + p_\theta \dot{\theta} + p_\varphi \dot{\varphi} = \frac{p_r^2}{m} + \frac{p_\theta^2}{m r^2} + \frac{p_\varphi^2}{m r^2 \sin^2 \theta} = \frac{1}{m} \left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2 \theta} \right) \quad (2.2)$$

$$\begin{aligned} T &= \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\varphi}^2 \right) = \\ &= \frac{1}{2} m \left(\frac{p_r^2}{m^2} + r^2 \frac{p_\theta^2}{m^2 r^4} + r^2 \sin^2 \theta \frac{p_\varphi^2}{m^2 r^4 \sin^2 \theta} \right) \\ &= \frac{1}{2m} \left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2 \theta} \right) \quad (2.3) \end{aligned}$$

By combining Eqs. (2.1) - (2.3), we find that

$$H = \frac{1}{m} \left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2 \theta} \right) - \frac{1}{2m} \left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2 \theta} \right) + V$$

$$= \frac{1}{2m} \left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2 \theta} \right) + V \quad (2.4)$$

From Eq. (14) and Eq. (2.3), we immediately obtain,

$$H = T + V = \frac{1}{2m} \left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2 \theta} \right) + V,$$

which is identical to Eq. (2.4).

Eqs. (13) and (14) must give the same result, since

- Eqs. (1) - (3) defining the spherical coordinates do not depend on time
 $\left(\frac{\partial x}{\partial t} = \frac{\partial y}{\partial t} = \frac{\partial z}{\partial t} = 0 \right)$

- We consider here a force that can be derived from the potential $V(\vec{r}, t)$.

In the class, we have learnt that if the above two conditions are satisfied, H equals $T + V$.

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3. (a)
$$V(x) = V(x_0) - \int_{x_0}^x F_x(x') dx'$$

Let us choose $x_0 = 0$, so that

we can write $\parallel 0$

$$V(x) = V(x=0) - \int_0^x F_x(x') dx'$$

$$= - \int_0^x F_x(x') dx' =$$

$$= +k \int_{-}^x x' dx'.$$

We obtain,

$$V(x) = k \frac{(x')^2}{2} \bigg|_{x'=0}^{x'=x} = \frac{1}{2} k x^2.$$

$$(b) \quad L(x, \dot{x}, t) = T(\dot{x}) - V(x, t)$$

$$T(\dot{x}) = \frac{1}{2} m \dot{x}^2$$

$$V(x, t) \equiv V(x) = \frac{1}{2} k x^2$$

$$L = L(x, \dot{x}, t) = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

Since there are no constraints and a force can be derived from $V(x)$,

$$H = T + V.$$

We obtain,

$$p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} = m \dot{x} \Rightarrow$$

$$\Rightarrow \dot{x} = \frac{p_x}{m}$$

Thus,

$$\begin{aligned} T &= \frac{1}{2} m \dot{x}^2 = \frac{1}{2} m \cdot \frac{p_x^2}{m^2} = \\ &= \frac{1}{2m} p_x^2, \end{aligned}$$

so that

$$\begin{aligned} H &= H(x, p_x, t) = \frac{1}{2m} p_x^2 + \\ &+ \frac{1}{2} k x^2 \end{aligned}$$

(c) The Lagrange equation has the form,

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = 0,$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} = m \dot{x},$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} = m \ddot{x}$$

$$\frac{\partial \mathcal{L}}{\partial x} = - \frac{\partial V}{\partial x} = - \frac{1}{2} k \cdot (2x) = -kx.$$

Thus,

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = m \ddot{x} + kx = 0,$$

so that

$$m \ddot{x} = -kx = F_x$$

(d) For ODEs of the general form,

$$a \ddot{x} + b \dot{x} + cx = 0 \quad (a, b, c - \text{constants}),$$

we always first solve the characteristic equation,

$$a \lambda^2 + b \lambda + c = 0, \text{ and}$$

use roots $\lambda_{1,2}$ to construct the

general solution. If both λ_1 and λ_2 are real and $\lambda_1 \neq \lambda_2$, we have

$$x(t) = A e^{\lambda_1 t} + B e^{\lambda_2 t} \quad (A, B - \text{constants}).$$

If λ_1 and λ_2 are complex,

$$\lambda_{1,2} = \alpha \pm \beta i,$$

then

$$x(t) = e^{\alpha t} (A \cos \beta t + B \sin \beta t)$$

In our case, our ODE has the form,

$$m \ddot{x} + kx = 0.$$

The characteristic equation,

$$m \lambda^2 + k = 0,$$

has two roots,

$$\lambda_{1,2} = \pm \sqrt{\frac{k}{m}} i = \pm i \omega,$$

where ω is the angular frequency.

Thus, the general solution of Newton's equation is

$$x(t) = A \cos \omega t + B \sin \omega t,$$

$$x(t=0) = 0 \text{ implies that}$$

$$A = 0,$$

$$\dot{x}(t) = -\omega A \sin \omega t + B \omega \cos \omega t,$$

so that

$$\dot{x}(t=0) = B \omega = v_0,$$

and

$$B = \frac{v_0}{\omega},$$

In consequence,

$$x(t) = \frac{v_0}{\omega} \sin \omega t$$

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Now,

$$T + V = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2.$$

$$\dot{x}(t) = \frac{v_0}{\cancel{\omega}} \cancel{\omega} \cos \omega t = v_0 \cos \omega t$$

$$\frac{1}{2} m \dot{x}^2 = \frac{1}{2} m v_0^2 \cos^2 \omega t$$

$$\therefore \frac{1}{2} k x^2 = \frac{1}{2} k \frac{v_0^2}{\omega^2} \sin^2 \omega t$$

$$= \frac{1}{2} m \cancel{\omega^2} \frac{v_0^2}{\cancel{\omega^2}} \sin^2 \omega t$$

$$= \frac{1}{2} m v_0^2 \sin^2 \omega t.$$

Thus,

$$T + V = \frac{1}{2} m v_0^2 \cos^2 \omega t +$$

$$+ \frac{1}{2} m v_0^2 \sin^2 \omega t = \frac{1}{2} m v_0^2.$$

$T + V$ is a constant and the value of

$T + V$ is

$$\frac{1}{2} m v_0^2.$$

For example,

- at $t=0$, $T = \frac{1}{2} m v_0^2$, $V = 0$, $T+V = \frac{1}{2} m v_0^2$
- at $t = \frac{\pi}{2\omega} = \frac{\pi}{4\nu} = \frac{T}{4}$ (ν -frequency, T -period),
 $T = 0$, $V = \frac{1}{2} m v_0^2$, $T+V = \frac{1}{2} m v_0^2$

4. According to Eq. (22),

$$U = q\phi - \frac{q}{c} \vec{A} \cdot \vec{v} =$$

$$= q\phi - \frac{q}{c} (A_x \dot{x} + A_y \dot{y} + A_z \dot{z}).$$

Let us calculate $\frac{\partial U}{\partial \dot{x}}$ and $\frac{\partial U}{\partial x}$.

$$\frac{\partial U}{\partial \dot{x}} = -\frac{q}{c} A_x \quad (4.1)$$

(recall that ϕ and $\vec{A}$ depend on x, y, z , and t and that they do not depend on $\dot{x}, \dot{y}$, and $\dot{z}$),

$$\frac{\partial U}{\partial x} = q \frac{\partial \phi}{\partial x} - \frac{q}{c} \left(\dot{x} \frac{\partial A_x}{\partial x} + \dot{y} \frac{\partial A_y}{\partial x} + \dot{z} \frac{\partial A_z}{\partial x} \right) \quad (4.2)$$

From Eq. (4.1) we obtain,

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial U}{\partial \dot{x}} \right) &= - \frac{q}{c} \frac{dA_x}{dt} = \\ &= - \frac{q}{c} \left(\frac{\partial A_x}{\partial x} \dot{x} + \frac{\partial A_x}{\partial y} \dot{y} + \frac{\partial A_x}{\partial z} \dot{z} + \frac{\partial A_x}{\partial t} \right) \quad (4.3) \end{aligned}$$

Thus,

$$\begin{aligned} - \frac{\partial U}{\partial x} + \frac{d}{dt} \left(\frac{\partial U}{\partial \dot{x}} \right) &= - q \frac{\partial \phi}{\partial x} \\ &+ \frac{q}{c} \left(\dot{x} \frac{\partial A_x}{\partial x} + \dot{y} \frac{\partial A_y}{\partial x} + \dot{z} \frac{\partial A_z}{\partial x} \right) \end{aligned}$$

$$\begin{aligned}
 & -\frac{q}{c} \left(\dot{x} \frac{\partial A_x}{\partial x} + \dot{y} \frac{\partial A_x}{\partial y} + \dot{z} \frac{\partial A_x}{\partial z} + \frac{\partial A_x}{\partial t} \right) \\
 & = -q \frac{\partial \phi}{\partial x} - \frac{q}{c} \left(-\dot{x} \frac{\partial A_x}{\partial x} - \dot{y} \frac{\partial A_y}{\partial x} \right. \\
 & \quad \left. - \dot{z} \frac{\partial A_z}{\partial x} + \dot{x} \frac{\partial A_x}{\partial x} + \dot{y} \frac{\partial A_x}{\partial y} + \dot{z} \frac{\partial A_x}{\partial z} \right. \\
 & \quad \left. + \frac{\partial A_x}{\partial t} \right) \\
 & = -q \frac{\partial \phi}{\partial x} - \frac{q}{c} \left[-\dot{y} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \right. \\
 & \quad \left. + \dot{z} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \frac{\partial A_x}{\partial t} \right] \quad (4.4)
 \end{aligned}$$

Recall that

$$\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = (\vec{\nabla} \times \vec{A})_z,$$

$$\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = (\vec{\nabla} \times \vec{A})_y,$$

so that Eq. (4.4) becomes,

$$-\frac{\partial \mathcal{U}}{\partial x} + \frac{d}{dt} \left(\frac{\partial \mathcal{U}}{\partial \dot{x}} \right) = q \left(-\frac{\partial \phi}{\partial x} - \frac{1}{c} \frac{\partial A_x}{\partial t} \right) - \frac{q}{c} \left[-\dot{y} (\vec{\nabla} \times \vec{A})_z + \dot{z} (\vec{\nabla} \times \vec{A})_y \right].$$

From Eqs. (20) and (21),

$$-\frac{\partial \phi}{\partial x} - \frac{1}{c} \frac{\partial A_x}{\partial t} = E_x \text{ and}$$

$$\vec{\nabla} \times \vec{A} = \vec{B}, \text{ so that}$$

$$-\frac{\partial \mathcal{U}}{\partial x} + \frac{d}{dt} \left(\frac{\partial \mathcal{U}}{\partial \dot{x}} \right) = q E_x +$$

$$+ \frac{q}{c} (\dot{y} B_z - \dot{z} B_y)$$

$$= q E_x + \frac{q}{c} (\vec{\nabla} \times \vec{B})_x = F_x,$$

where F_x is the x-component of

$\vec{F}$, Eq. (19).

In consequence,

$$F_x = - \frac{\partial U}{\partial x} + \frac{d}{dt} \left(\frac{\partial U}{\partial \dot{x}} \right),$$

which is a desired result, Eq. (23)

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Note: Similar arguments would allow us to prove that Eq. (23) is also true for the y and z components of $\vec{F}$.