

1. (4 pts.) Consider the classical particle with mass m . Let x , y , and z define the Cartesian components of the radius vector[#] $\mathbf{r}$ from the origin to the particle and let r , θ , and ϕ designate the spherical components of $\mathbf{r}$, so that

$$x = r \sin \theta \cos \phi, \quad (1)$$

$$y = r \sin \theta \sin \phi, \quad (2)$$

$$z = r \cos \theta. \quad (3)$$

Derive the formula for the kinetic energy of the particle m ,

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2). \quad (4)$$

Recall that the dot above a symbol designates the time derivative $\frac{d}{dt}$.

2. (8 pts.) Consider the classical particle with mass m under the influence of a force derived from the potential $V(\mathbf{r}, t)$ (no constraints). You have learned in class that the canonical momenta p_ℓ , $\ell = 1, 2, 3$, are related to Cartesian components of the linear momentum $\mathbf{p} = m\dot{\mathbf{r}}$ as follows:

$$p_\ell = p_x \frac{\partial x}{\partial q_\ell} + p_y \frac{\partial y}{\partial q_\ell} + p_z \frac{\partial z}{\partial q_\ell}. \quad (5)$$

Let r , θ , and ϕ designate the spherical components of $\mathbf{r}$, so that $q_1 = r$, $q_2 = \theta$, and $q_3 = \phi$.

- (a) Using the general definition of the canonical momentum p_ℓ ,

$$p_\ell = \frac{\partial L}{\partial \dot{q}_\ell}, \quad L = T - V, \quad (6)$$

and the fact that the kinetic energy T expressed in terms of r , θ , and ϕ is given by Eq. (4) (see Problem 1), prove that

$$p_r = m\dot{r}, \quad (7)$$

$$p_\theta = mr^2\dot{\theta}, \quad (8)$$

$$p_\phi = mr^2\sin^2\theta\dot{\phi}. \quad (9)$$

- (b) Show that

$$p_r = \mathbf{p} \cdot \frac{\mathbf{r}}{r}, \quad (10)$$

so that p_r has an interpretation of a linear momentum along the radial direction.

[#]) Bold symbols designate vector quantities.

- (c) Show that p_ϕ can be interpreted as the z -component of the angular momentum vector

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}. \quad (11)$$

In other words, L_z is conjugate to ϕ . *Hint:* Note that

$$L_z = xp_y - yp_x = m(x\dot{y} - y\dot{x}), \quad (12)$$

and use the relationships (1)–(3) and the chain rule to express $\dot{x}$, $\dot{y}$, and $\dot{z}$ in terms of $\dot{r}$, $\dot{\theta}$, and $\dot{\phi}$.

- (d) By using the equations presented in (a) and (b) and the definition of the Hamiltonian,

$$H = \sum_{\ell=1}^3 p_\ell v_\ell - L, \quad (13)$$

where $L = T - V$ is the Lagrangian and $v_\ell = \dot{q}_\ell$ (recall that v_ℓ has to be expressed in terms of q_ℓ and p_ℓ , $\ell = 1, 2, 3$), show that

$$H = \frac{1}{2m} \left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2 \theta} \right) + V. \quad (14)$$

Show that the same expression for H is obtained by using the formula

$$H = T + V. \quad (15)$$

Explain why Eqs. (13) and (15) lead to the same result.

3. (8 pts.) The force that governs the motion of a harmonic oscillator with mass m and the angular frequency ω is defined as

$$F_x = -kx, \quad (16)$$

where $k = m\omega^2$.

- (a) Calculate the potential function V [assume that $V(x = 0) = 0$].
 (b) Calculate the Lagrangian and the Hamiltonian.
 (c) Show that the Lagrange equation reduces in this case to ordinary Newton's equation

$$m\ddot{x} = F_x, \quad (17)$$

where F_x is defined by Eq. (16).

- (d) Solve Newton's equation [Eq. (17)], i.e. determine $x = x(t)$, assuming the following initial conditions: $x(t = 0) = 0$, $\dot{x}(t = 0) = v_0$. Use the solution $x(t)$, the definition of the kinetic energy T , and the formula for the potential function V [see item (a) above] to show that

$$T + V = \text{const.} \quad (18)$$

What is the value of $T + V$ in this case?

4. (5 pts.) The force on a particle with charge q is given by the well-known Lorentz formula

$$\mathbf{F} = q \left[\mathbf{E} + \frac{1}{c}(\mathbf{v} \times \mathbf{B}) \right], \quad (19)$$

where

$$\mathbf{E} = -\nabla\phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \quad (20)$$

and

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (21)$$

are, respectively, the electric and magnetic field vectors, ϕ is the electric potential, $\mathbf{A}$ is the magnetic vector potential, and $\mathbf{v} = (\dot{x}, \dot{y}, \dot{z})$ is the velocity. The velocity-dependent potential U corresponding to Lorentz force $\mathbf{F}$, Eq. (19), is given by the formula,

$$U = q\phi - \frac{q}{c} \mathbf{A} \cdot \mathbf{v}. \quad (22)$$

Show that the following equation,

$$F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt} \frac{\partial U}{\partial \dot{x}}, \quad (23)$$

is true.

Hint: Calculate $\frac{\partial U}{\partial x}$ and $\frac{\partial U}{\partial \dot{x}}$ using formula (22) and compare the result with the formula for the x component of $\mathbf{F}$ resulting from Eqs. (19)–(21). Recall that for any vector field $\mathbf{G} = \mathbf{G}(x, y, z, t)$,

$$(\nabla \times \mathbf{G})_x = \frac{\partial G_z}{\partial y} - \frac{\partial G_y}{\partial z}, \quad (24)$$

$$(\nabla \times \mathbf{G})_y = \frac{\partial G_x}{\partial z} - \frac{\partial G_z}{\partial x}, \quad (25)$$

$$(\nabla \times \mathbf{G})_z = \frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y}. \quad (26)$$