OVERVIEW OF CLASSICAL MECHANICS AND ELECTRODYNAMICS#

- **Newton's equation for one particle in one dimension**

$$F_x \equiv F_x(x, \dot{x}, t) = m\ddot{x} \equiv m\frac{d^2x}{dt^2}$$

- **Newton's equations for one particle in three dimensions**

$$\mathbf{F} \equiv \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t) = m\ddot{\mathbf{r}} \equiv m\frac{d^2\mathbf{r}}{dt^2}$$

or

$$F_x = m\ddot{x} \equiv m\frac{d^2x}{dt^2}, \quad F_y = m\ddot{y} \equiv m\frac{d^2y}{dt^2}, \quad F_z = m\ddot{z} \equiv m\frac{d^2z}{dt^2}$$

- **Mechanics of a system of N particles without constraints according to Newton**

$$\mathbf{F}_i \equiv \mathbf{F}_i(\mathbf{r}_1, \ldots, \mathbf{r}_N, \dot{\mathbf{r}}_1, \ldots, \dot{\mathbf{r}}_N, t) = m_i\ddot{\mathbf{r}}_i, \quad i = 1, \ldots, N$$

or

$$X_j \equiv X_j(x, \dot{x}, t) = m_j\ddot{x}_j, \quad j = 1, \ldots, 3N, \quad x \equiv (x_1, \ldots, x_{3N}), \quad \dot{x} \equiv (\dot{x}_1, \ldots, \dot{x}_{3N})$$

$$\mathbf{F}_i = \mathbf{F}_i^{(\mathrm{ext})} + \sum_{j=1}^{N} \mathbf{F}_{ji}, \quad \mathbf{F}_{ji} = -\mathbf{F}_{ij} \quad (\mathbf{F}_{ii} = 0)$$

#) Bold symbols designate vector quantities.

- **Newtonian momentum for one particle in one dimension**

$$p = mv = m\dot{x}$$

- **Newtonian momentum for one particle in three dimensions**

$$\mathbf{p} = m\mathbf{v} = m\dot{\mathbf{r}}$$

or

$$p_x = m\dot{x}, \;\; p_y = m\dot{y}, \;\; p_z = m\dot{z}$$

- **Newtonian momenta for a system of N particles**

$$p_j = m_j \dot{x}_j, \quad j = 1, \ldots, 3N,$$

- **Kinetic energy (T) for one particle in one dimension**

$$T = \frac{1}{2}m\dot{x}^2 = \frac{p^2}{2m}, \qquad p = m\dot{x}$$

- **Kinetic energy (T) for one particle in three dimensions**

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2), \qquad p_x = m\dot{x}, \;\; p_y = m\dot{y}, \;\; p_z = m\dot{z}$$

- **Kinetic energy (T) for N particles**

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) = \sum_{i=1}^{N} \frac{1}{2m_i} (p_{x,i}^2 + p_{y,i}^2 + p_{z,i}^2)$$

or

$$T = \tfrac{1}{2} \sum_{j=1}^{3N} m_j (\dot{x}_j)^2 = \tfrac{1}{2} \sum_{j=1}^{3N} \frac{p_j^2}{2m_j}$$

- **Potential function (V) for one particle in one dimension**

$$F_x = -\frac{\partial V(x,t)}{\partial x}$$

- **Potential function (V) for one particle in three dimensions**

$$\mathbf{F} = -\boldsymbol{\nabla} V(\mathbf{r}, t), \quad \boldsymbol{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

or

$$F_x = -\frac{\partial V(x,y,z,t)}{\partial x}, \quad F_y = -\frac{\partial V(x,y,z,t)}{\partial y}, \quad F_z = -\frac{\partial V(x,y,z,t)}{\partial z}$$

- **Potential function (V) for N particles without constraints**

$$\mathbf{F}_i = -\boldsymbol{\nabla}_i V(\mathbf{r}_1, \ldots, \mathbf{r}_N, t), \quad \boldsymbol{\nabla}_i = \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}, \frac{\partial}{\partial z_i} \right)$$

or

$$F_{x,i} = -\frac{\partial}{\partial x_i} V(x_1, y_1, z_1, \ldots, x_N, y_N, z_N, t)$$
$$F_{y,i} = -\frac{\partial}{\partial y_i} V(x_1, y_1, z_1, \ldots, x_N, y_N, z_N, t)$$
$$F_{z,i} = -\frac{\partial}{\partial z_i} V(x_1, y_1, z_1, \ldots, x_N, y_N, z_N, t)$$

or

$$X_j = -\frac{\partial}{\partial x_j} V(x, t)$$

$$X_j(x, \dot{x}, t) = m_j \ddot{x}_j, \quad j = 1, \ldots, 3N, \quad x \equiv (x_1, \ldots, x_{3N}),$$

When V does not depend on time (t) explicitly, we call V the *potential energy* and we call forces $\mathbf{F}_i$ the *conservative forces.* For the conservative forces, the sum of the kinetic and potential energies remains constant during the motion.

- **Angular momentum for one particle**

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}, \quad \mathbf{p} = m\mathbf{v} = m\dot{\mathbf{r}}$$

- **Angular momentum for N particles**

$$\mathbf{L} = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{p}_i, \quad \mathbf{p}_i = m_i \mathbf{v}_i = m\dot{\mathbf{r}}_i$$

- **Mechanics of a system of particles with and without constraints (holonomic constraints)**

$$f_p(x_1,\dots,x_{3N},t)=0,\quad p=1,\dots,k\ (k\le 3N)$$

$$X_j+X_j^{(\mathrm{c})}=m_j\ddot{x}_j,\quad j=1,\dots,3N$$

$$\sum_{j=1}^{3N}X_j^{(\mathrm{c})}\,\delta x_j=0,\ \text{where}\ \sum_{j=1}^{3N}\frac{\partial f_p}{\partial x_j}\,\delta x_j=0$$

- **D'Alembert's principle**

$$f_p(x_1,\dots,x_{3N},t)=0,\quad p=1,\dots,k\ (k\le 3N)$$

$$\sum_{j=1}^{3N}(X_j-m_j\ddot{x}_j)\delta x_j=0,\ \text{where}\ \sum_{j=1}^{3N}\frac{\partial f_p}{\partial x_j}\,\delta x_j=0$$

- **Generalized and curvilinear coordinates for systems with and without constraints**

$$\begin{aligned}x_i&=x_i(q_1,\dots,q_f,t),\\ y_i&=y_i(q_1,\dots,q_f,t),\\ z_i&=z_i(q_1,\dots,q_f,t),\\ i=1,\dots,N,&\quad f=3N-k\end{aligned}$$

or

$$x_j=x_j(q_1,\dots,q_f,t),\quad j=1,\dots 3N,\quad f=3N-k$$

Example: spherical coordinates for one particle without constraints ($N = 1$, $k = 0$, $f = 3$)

$$x = r \sin\vartheta \cos\varphi, \quad y = r \sin\vartheta \sin\varphi, \quad z = r \cos\vartheta$$

Example: polar coordinates for one particle moving on the (x, y) plane ($N = 1$, $k = 1$, $f = 2$)

$$x = r \cos\varphi, \quad y = r \sin\varphi, \quad z = 0$$

- **Generalized coordinates, D'Alembert's principle in generalized coordinates**

$$x_j = x_j(q_1, \ldots, q_f, t), \quad j = 1, \ldots 3N, \quad f = 3N - k$$

or

$$x = x(q, t), \quad q \equiv (q_1, \ldots, q_f)$$

$$\sum_{\ell=1}^{f} \left[Q_\ell - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\ell}\right) + \frac{\partial T}{\partial q_\ell} \right] \delta q_\ell = 0$$

$$Q_\ell = \sum_{j=1}^{3N} X_j \frac{\partial x_j}{\partial q_\ell}, \quad T = \frac{1}{2} \sum_{j=1}^{3N} m_j (\dot{x}_j)^2, \quad \delta q_\ell \text{ arbitrary}$$

- **Lagrange's equations**

$$Q_\ell = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\ell}\right) - \frac{\partial T}{\partial q_\ell}, \quad \ell = 1, \ldots, f$$

- **Lagrange's equations for the forces that can be obtained from a potential function** V ($X_j = -\frac{\partial V}{\partial x_j}$ or $Q_\ell = -\frac{\partial V}{\partial q_\ell}$, where $V = V(x,t)$)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\ell}\right) - \frac{\partial L}{\partial q_\ell} = 0, \quad \ell = 1, \ldots, f$$

$$L = L(q, \dot{q}, t) = T(\dot{x}(q, \dot{q}, t)) - V(x(q,t), t)$$

$$q \equiv (q_1, \ldots, q_f), \quad \dot{q} \equiv (\dot{q}_1, \ldots, \dot{q}_f)$$

- **Lagrange's equations for the forces that can be obtained from a generalized (velocity-dependent) potential function** U ($X_j = -\frac{\partial U}{\partial x_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{x}_j}\right)$ or $Q_\ell = -\frac{\partial U}{\partial q_\ell} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_\ell}\right)$, where $U = U(x, \dot{x}, t)$)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\ell}\right) - \frac{\partial L}{\partial q_\ell} = 0, \quad \ell = 1, \ldots, f$$

$$L = L(q, \dot{q}, t) = T(\dot{x}(q, \dot{q}, t)) - U(x(q,t), \dot{x}(q, \dot{q}, t), t)$$

- **Lagrange's equations in terms of conjugate momenta**

$$p_\ell \equiv \frac{\partial L}{\partial \dot{q}_\ell}$$

$$\dot{p}_\ell = \frac{\partial L}{\partial q_\ell}, \quad \ell = 1, \ldots, f$$

- **The Hamilton equations**

$$\dot{q}_\ell = \frac{\partial H}{\partial p_\ell}, \quad \dot{p}_\ell = -\frac{\partial H}{\partial q_\ell}, \quad \ell = 1, \ldots, f$$

$$H = H(q,p,t) = \sum_{\ell=1}^{f} p_l\, v_l(q,p,t) - L(q, v(q,p,t), t)$$

$$q \equiv (q_1, \ldots, q_f), \quad p \equiv (p_1, \ldots, p_f), \quad v \equiv (v_1, \ldots, v_f)$$

$v_\ell = \dot{q}_\ell$ and $v_\ell(q,p,t)$ is obtained by solving the system of equations

$$p_\ell = \frac{\partial L(q,v,t)}{\partial v_\ell}, \quad \ell = 1, \ldots, f$$

for v_ℓ.

For contraints that do not depend on time ($\frac{\partial x_j(q,t)}{\partial t} = 0$ or $x_j = x_j(q)$) and for the forces that are obtained from a potential function V, we have

$$H(q,p,t) = T(q, v(q,p)) + V(q,t)$$

- **The equations of motion in the Poisson bracket notation**

$$F = F(q,p,t), \quad G = G(q,p,t)$$

$$\{F, G\} = \sum_{\ell=1}^{f} \left(\frac{\partial F}{\partial q_\ell}\frac{\partial G}{\partial p_\ell} - \frac{\partial F}{\partial p_\ell}\frac{\partial G}{\partial q_\ell} \right)$$

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \{F, H\}$$

- **Maxwell's equations for the electric and magnetic fields, E and B, respectively, generated by charge and current distributions, ρ and j, respectively. The Lorentz force formula for a particle moving in the electromagnetic field**

$$\mathbf{E} = \mathbf{E}(x, y, z, t), \quad \mathbf{B} = \mathbf{B}(x, y, z, t)$$

$$\rho = \rho(x, y, z, t), \quad \mathbf{j} = \mathbf{j}(x, y, z, t)$$

$$\boldsymbol{\nabla} \bullet \mathbf{E} = 4\pi\rho \qquad \text{(Gauss' law)}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \qquad \text{(Faraday's law)}$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{j} \qquad \left(\frac{\partial \mathbf{E}}{\partial t} = 0 \rightarrow \text{Ampere's law}\right)$$

$$\boldsymbol{\nabla} \bullet \mathbf{B} = 0 \qquad \text{(magnetic charges do not exist)}$$

$$\mathbf{F} = q\left[\mathbf{E} + \frac{1}{c}(\mathbf{v} \times \mathbf{B})\right] \qquad \text{(the Lorentz force)}$$

Notation:

$$\boldsymbol{\nabla} \bullet \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$(\boldsymbol{\nabla} \times \mathbf{A})_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \quad (\boldsymbol{\nabla} \times \mathbf{A})_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \quad (\boldsymbol{\nabla} \times \mathbf{A})_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$$

- **Scalar and vector potentials, generalized potential for electromagnetic forces**

$$\phi = \phi(x, y, z, t), \quad \mathbf{A} = \mathbf{A}(x, y, z, t)$$

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$$

$$U = q\phi - \frac{q}{c}\mathbf{A} \bullet \mathbf{v}$$

- **Langrangian and Hamiltonian for a particle in an electromagnetic field**

$$L = T - q\phi + \frac{q}{c}\mathbf{A} \bullet \mathbf{v}$$

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right)^2 + q\phi$$

- **Wave equation in three dimensions**

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} = 0, \quad \psi = \psi(x, y, z, t)$$

- **Plane waves (in one dimension)**

$$\psi(x,t) = Ae^{i(kx - \omega t)} \text{ or } \psi(x,t) = Ae^{i(kx-\omega t)} + Be^{-i(kx-\omega t)} \text{ or } \psi(x,t) = C\sin(kx - \omega t + \gamma)$$

$$(k = 2\pi/\lambda,\ \omega = 2\pi\nu,\ v = \lambda\nu = \omega/k)$$