Week 4: Chap. 3 Statistics of Radioactivity

MICHIGAN STATE UNIVERSITY

Vacuum Technology

General use of Statistical Distributions in Radiation Measurements

-- Fluctuations in Number

--- distribution function models

-- Fluctuations in Time

General Detector Properties

"Without deviation from the norm, progress is not possible."
Frank Zappa, 1986

MICHIGAN STATE UNIVERSITY

Fact: Randomness of decay leads to statistical fluctuations.

Goals:

(I) <u>Characterization of fluctuations in data</u>: The problem is: A set of measurements of one quantity contains a range of answers. How should we describe the *set* ? [could enumerate a small set: {7,12,15,11}, 11.25 +/- 3.3]

Experimental Mean: $\overline{x_e} = \frac{1}{N}\sum_{1}^{N} x_i$

Experimental Residuals: $d_i = x_i - \overline{x_e} \quad note \quad \sum_{1}^{N} d_i = 0$

Deviation from (*the*) mean And (*the*) variance: $\varepsilon_i = x_i - \overline{x} \qquad \sigma^2 = \overline{\varepsilon^2} = \frac{1}{N}\sum_{1}^{N}\left(x_i - \overline{x}\right)^2$

Expt. Sample Variance: $s^2 = \frac{1}{N-1}\sum_{1}^{N}\left(\overline{x_e} - x_i\right)^2 \quad practical\ expresssion \quad s^2 \approx \overline{x_i^2} - \left(\overline{x_e}\right)^2$

MICHIGAN STATE UNIVERSITY

(II) Fluctuation Models:

A) The binomial distribution, a theory with three general parameters (a,b,n)

$$(a+b)^n = a^n / 0! + (na^{n-1}\, b^1)/1! + [n(n-1)\, a^{n-2}\, b^2\,]/\, 2! + \ldots\, [n!\, a^{n-x}\, b^x\,]/\, (n-x)!x! + b^n / 0!$$

Special case of a fractional probability, p, which allows one variable to be removed,
$b = p$, $a = 1-p$ so that $a+b = 1$ and automatically normalizes the distribution:

$$(a+b)^n = \Sigma\, P_n\,(x) = (1-p)^n + \ldots\, [n!\, (1-p)^{n-x}\, p^x\,]/\, (n-x)!x! \;\ldots + p^n$$

Thus, if "p" is the probability of success and "n" is the number of tries, then x can be seen to be the number of successes.

Example 1: Coin flip: $p = ½$ $(1-p) =1/2$

Example 2: Toss die: $p = 1/6$ $(1-p) = 5/6$

Example 3: (fractional) Probability of radioactive decay:

$$p = (\#\text{ decay}) / (\#\text{ sample}) = (N_0 - N) / N_0 = (1 - e^{-\lambda t}) \;\; ; \;\; (1-p) = e^{-\lambda t}$$

(II) Fluctuation Models:

A) Binomial distribution function (theory),

special case: n tries with p fractional probability, gives x successes

$P_n(x) = [n! / (n-x)!x!] \; (1-p)^{n-x} \; p^x$

Mean = p*n, $\sigma^2 = p\, n\, (1-p)$, $\sigma^2/\text{mean} = (1-p)$

Need to know p & n to evaluate each element of this distribution.

Question: Toss three dice all at once, what is chance of getting a "5" ?

(the same as three throws of one die to get a "5" … an experiment beckons)

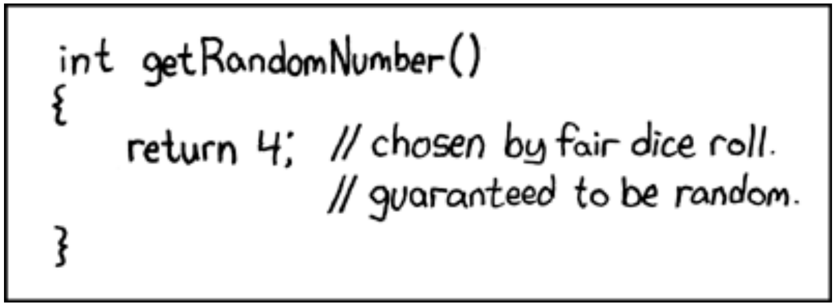
n=3, x is number of 5's

x = 0	$P_3(0) = [3! / (3-0)!\,0!]\,(1/6)^0\,(5/6)^3$	0.5787
1	$P_3(1) = [3! / (3-1)!\,1!]\,(1/6)^1\,(5/6)^2$	0.3472
2	$P_3(2) = [3! / (3-2)!\,2!]\,(1/6)^2\,(5/6)^1$	0.06944
3	$P_3(3) = [3! / (3-3)!\,3!]\,(1/6)^3\,(5/6)^0$	0.00463




Note: $\Sigma P_n(x) = 1$

Test random dice-roller app (https://xkcd.com/221)



Mean = p*n = 1/6 * 3 = 1/2

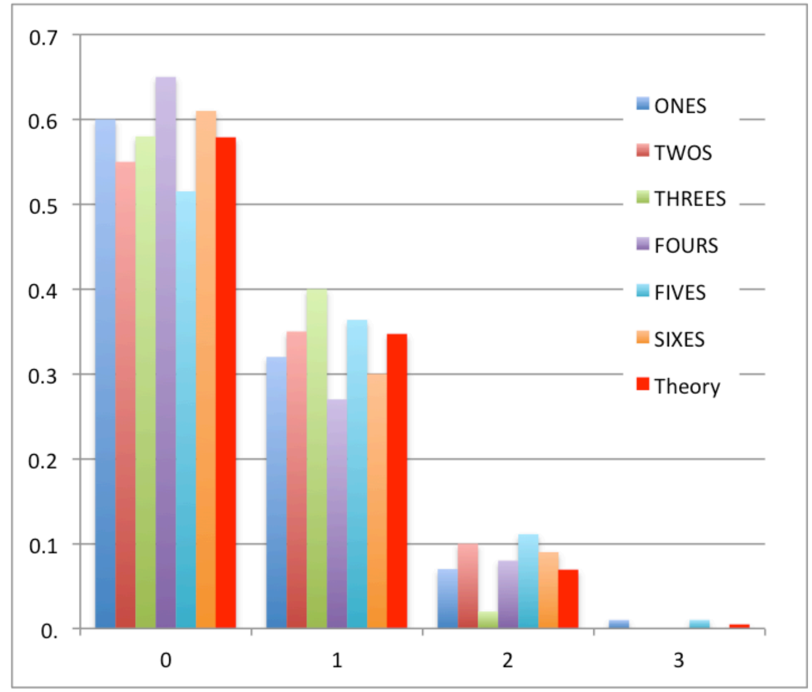
σ^2 = p n (1-p) = 1/6*3*(5/6) = 5/12

x = 0	P_3 (0) = [3! / (3-0)! 0!] $(1/6)^0$ $(5/6)^3$	0.5787
1	P_3 (1) = [3! / (3-1)! 1!] $(1/6)^1$ $(5/6)^2$	0.3472
2	P_3 (2) = [3! / (3-2)! 2!] $(1/6)^2$ $(5/6)^1$	0.06944
3	P_3 (3) = [3! / (3-3)! 3!] $(1/6)^3$ $(5/6)^0$	0.00463

Note: Σ P_n (x) = 1

Statistics & Measurement – Models –2a–

Test random dice-roller app



100 throws of 3 dice …

Mean = p*n = 1/6 * 3 = 1/2

σ^2 = p n (1-p) = 1/6*3*(5/6) = 5/12

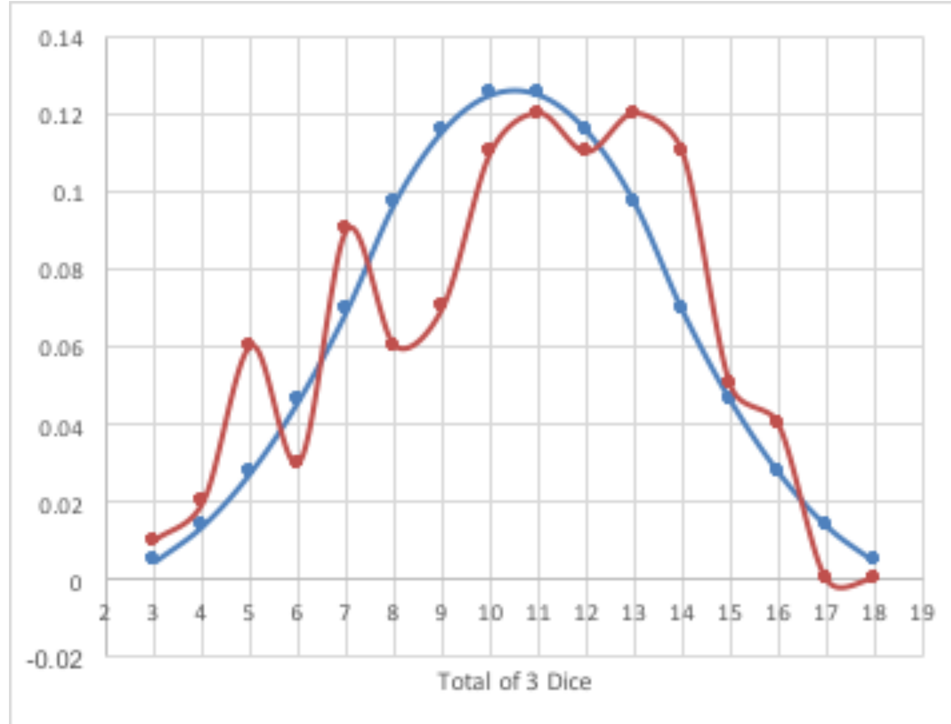
x = 0	P_3 (0) = [3! / (3-0)! 0!] $(1/6)^0$ $(5/6)^3$	0.5787
1	P_3 (1) = [3! / (3-1)! 1!] $(1/6)^1$ $(5/6)^2$	0.3472
2	P_3 (2) = [3! / (3-2)! 2!] $(1/6)^2$ $(5/6)^1$	0.06944
3	P_3 (3) = [3! / (3-3)! 3!] $(1/6)^3$ $(5/6)^0$	0.00463

Note: Σ P_n (x) = 1

Statistics & Measurement – Models –2a–

Test random dice-roller app

100 throws of 3 dice …



Mean = p*n = 1/6 * 3 = 1/2

σ^2 = p n (1-p) = 1/6*3*(5/6) = 5/12

x = 0	P_3 (0) = [3! / (3-0)! 0!] $(1/6)^0$ $(5/6)^3$	0.5787
1	P_3 (1) = [3! / (3-1)! 1!] $(1/6)^1$ $(5/6)^2$	0.3472
2	P_3 (2) = [3! / (3-2)! 2!] $(1/6)^2$ $(5/6)^1$	0.06944
3	P_3 (3) = [3! / (3-3)! 3!] $(1/6)^3$ $(5/6)^0$	0.00463

Note: $\Sigma P_n (x) = 1$

Statistics & Measurement – Models –3–

(II) Fluctuation Models:

B) The Poisson distribution ($p \to 0$) and large n (many tries before success)

$P_n(x) = [n!\,(1-p)^{n-x}\,p^x\,]/\,(n-x)!x! \to P_n(x) = [n^x\,(1-p)^{n-x}\,p^x\,]/x!$ Since $n!/n\text{-}x! \sim n^x$

$P_n(x) = [n^x\,p^x\,(1-p)^{n-x}]/x!$ rearrange

$$n^x p^x = (np)^x = \left(\bar{x}\right)^x \quad also \quad p = \bar{x}/n$$

$$P_n\left(x\right) = \frac{\left(\bar{x}\right)^x}{x!}\left(1 - \frac{\bar{x}}{n}\right)^{n-x}$$

$$when\ n >> x \quad P_{Poisson}\left(x\right) = \frac{\left(\bar{x}\right)^x}{x!}e^{-\bar{x}}$$

Only one parameter, the mean

$$\sum P_p(x) = 1 \qquad \bar{x} = pn \qquad \sigma^2 = \bar{x}\left(1-p\right) = \bar{x}$$

Asymmetric (theoretical) distribution with a tail on high side

MICHIGAN STATE UNIVERSITY

(II) Fluctuation Models:

C) Example of Poisson Distribution: Easy variant of the "birthday problem:" what is the probability that at least one person in this room has the same birthday as me?

$$P_{Poisson}(x) = \frac{\left(\bar{x}\right)^{x}}{x!} e^{-\bar{x}} \qquad n >> x$$

$$\bar{x} = pn \qquad p = \frac{1}{365.25}$$

$$P_P(x=0) = \frac{\left(\bar{x}\right)^{0}}{0!} e^{-\bar{x}}$$

$$P_P(x>0) = \sum_{x=1}^{n} P_P(x)$$

p =	0.002738	[1/365.25]
n=	92	
<x> = pn =	0.251882	
exp(-<x>) =	0.777336	
x	P(x)	
0	0.777336	
1	0.195797	
2	0.024659	
3	0.00207	
4	0.00013	
5	6.57E-06	
6	2.76E-07	

Statistics & Measurement – Models –4–

MICHIGAN STATE UNIVERSITY

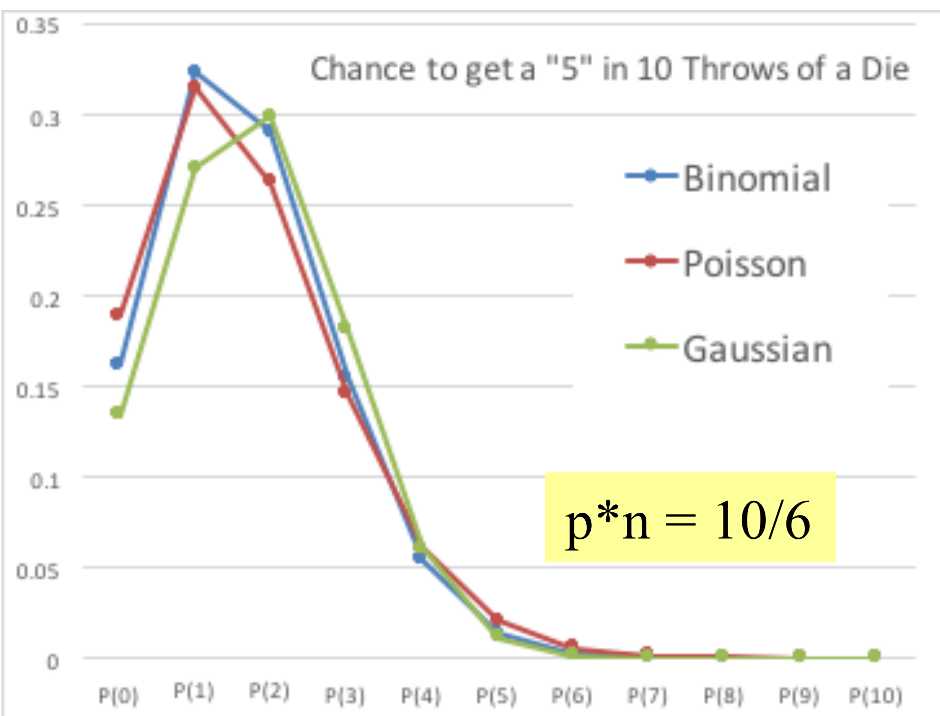
(II) Fluctuation Models:

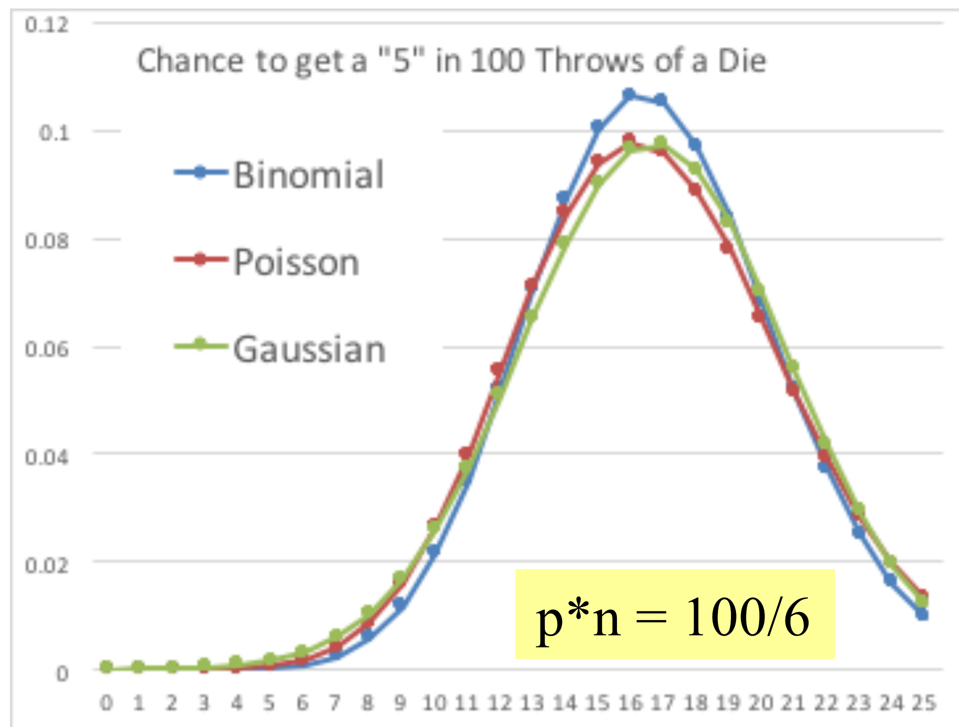
C) The Gaussian approximation, $p \rightarrow 0$, $n >> 1$ & mean = $p*n > 20$

$$P_G(x) = \frac{1}{\sqrt{2\pi \bar{x}}} e^{-(\bar{x}-x)^2/2\bar{x}} \quad or \quad G(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(\bar{x}-x)^2/2\sigma^2}$$

Symmetric Continuous distribution (one parameter) AKA chi-squared distribution:

$$G(\chi) = \frac{2}{\sqrt{2\pi}} e^{-\chi^2/2} \qquad where\ \chi = \left(\bar{x} - x\right)/\sigma$$





MICHIGAN STATE UNIVERSITY

(III) Analysis of data

A) Are fluctuations in a set of N-values consistent with statistical accuracy?

Measure: x_e and s^2 Model: mean gives σ^2 … thus the test: Is $s^2 = \sigma^2$?

Define a statistic χ in terms of measurable values that will provide a simple test.

$$\chi^2 = \sum_{i=1}^{N}\left(\frac{\overline{x_e} - x}{s}\right)^2 = \sum_{i=1}^{N}\frac{\left(\overline{x_e} - x\right)^2}{s^2} = \sum_{i=1}^{N}\frac{\left(\overline{x_e} - x\right)^2}{\overline{\overline{x_e}}}$$

(Bottom of Slide 1 today)

$$\chi^2 = \frac{\sum_{i=1}^{N}\left(\overline{x_e} - x\right)^2}{\overline{\overline{x_e}}} = \frac{s^2(N-1)}{\overline{\overline{x_e}}} \rightarrow \frac{\chi^2}{(N-1)} = \frac{s^2}{\overline{\overline{x_e}}}$$

But in a "good" data set with large N: $s^2 = \sigma^2 = \text{mean}$

$$\therefore \quad \frac{\chi^2}{(N-1)} = 1 \quad or \quad \chi^2 = (N-1)$$

Statistics & Measurement – Analysis –2–

MICHIGAN STATE UNIVERSITY

(III) Analysis of data

B) Only one measurement, estimate uncertainty?

There is only one measurement of an integer quantity, strictly speaking, you are out of luck.

However, one can posit that it must be the mean, and σ^2 = mean

One further assumes that the distribution is symmetric: x +/- σ

Fractional error: $\sigma / x = \sqrt{x} / x = 1 / \sqrt{x}$

(IV) Measurements in the presence of Background (itself measureable)

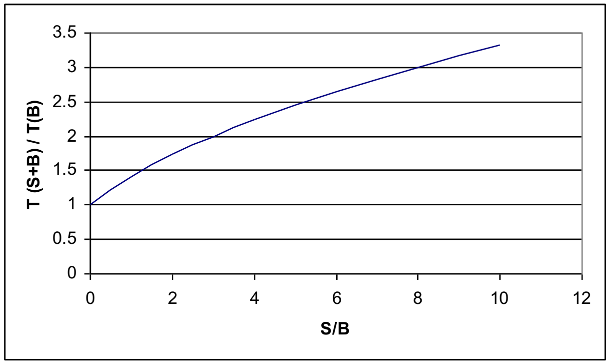
(1) measure background (B); (2) measure source + background (S+B).

S = (S+B) – B … nothing new here but there are two contributions to error.

How should the time be allocated to give minimum uncertainty in S ie s_S ?

Simplest case, constant rates the minimum in s_S comes when:

$$\frac{time_{S+B}}{time_B} = \sqrt{\frac{S+B}{B}} = \sqrt{\frac{S}{B}+1}$$

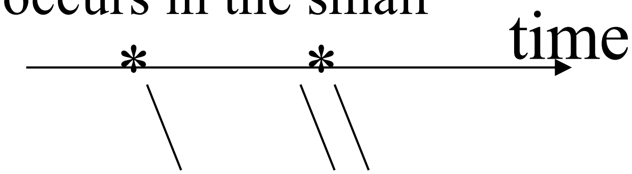


B diminishes in importance for large S

Statistics & Measurement – time distribution

(V) Time between random events, the *Interval distribution*:

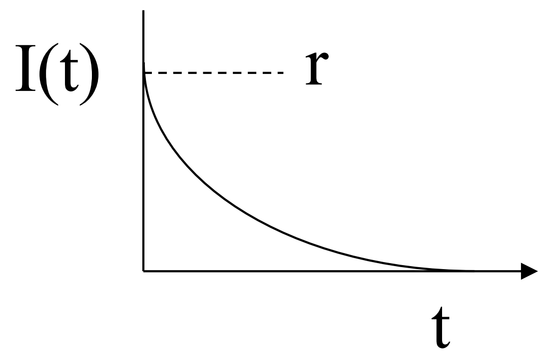
A) The product of the probability that nothing happens for some length of time, t, times the probability that the next event occurs in the small interval, dt.



$$I_1(t)dt = \left[P_P(x=0)\right]\left[r\ dt\right] \quad where \quad \text{r is average rate}$$

$$I_1(t)dt = \frac{\left(\bar{x}\right)^0}{0!} e^{-\bar{x}}\left[r\ dt\right] \quad where \quad \bar{x} = r * t$$

$$I_1(t)dt = e^{-r\,t} r\,dt$$



Note the most-probable time between events occurs at rt = 0 , note r > 0 …

The average time between events

$$\bar{t}_1 = \frac{\int_0^\infty t\, I_1(t)\,dt}{\int_0^\infty I_1(t)\,dt} = \frac{\int_0^\infty tr\,e^{-r\,t}\,dt}{1} = \frac{1}{r} \quad (r>0)$$

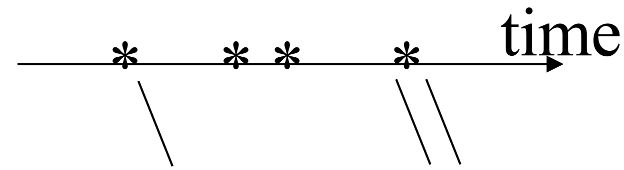
Statistics & Measurement – time distribution

(V) Time between random events, the *Interval distribution*:

B) Some people like to "scale-down" high counting rates by a factor, N. Similar arguments give the distribution of the N-th event as:

$$I_N(t)dt = P(N-1)\, r\, dt$$

$$I_N(t)dt = \frac{(rt)^{N-1}}{(N-1)!} e^{-rt} r dt$$



which is a curve that has a peak near the value (rt) = N-1

[find most-probable time from $dI_N/dt = 0$ … $t_{mp} = (N-1)/r$]

and an average time between scaled events:

$$\bar{t}_N = \frac{\int_0^\infty t\, I_N(t)\, dt}{\int_0^\infty I_N(t)\, dt} = \frac{N}{r} \quad (r > 0)$$

Statistics & Measurement – time distribution

MICHIGAN STATE UNIVERSITY

(V) Time between random events, the *Interval distribution*

C) Examples of Interval Distribution from nuclear physics:

1) One of the early results from the Sudbury Neutrino Observatory (SNO) was that 1968 charged-current neutrino events were observed in 306.4 days of operation. One concern on any given day in low counting rate experiments is that the detector has failed. What is the probability that no events were observed on any given day during this run?

Time passes until "evt occurs"

Probability of "event" in 1day

Probability of "not" in 1 day

2) The confirmation in 2013 of Element 115 claimed to have measured 30 atoms in a "three week run". Similarly, what is the probability that this experiment went 1 day without observing an event?

Week 4: Chap. 4 Basic Detector Principles

General use of Statistical Distributions

Basic Detector Principles

-- Current Mode
-- Pulse Mode
--- resolution
---- Fano Factor
--- efficiency
--- dead time

Ion Chambers