\[
\begin{align*}
\left( \text{Eq. (L-7)} \right) & \\
0 &= \phi \left( \frac{1}{\varepsilon} + \beta \right) + \phi \Delta \\
\text{becomes}
\end{align*}
\]

In these units, which are equivalent to Greenwich Mean Time, the coordinates in terms of the point radius \( r \) are expressed as \( \rho, \phi, \psi \). The energy is measured in multiples of the photon energy unit in terms of which the ground-state energy is defined. In the treatment of the ground state of the hydrogen atom, the energy \( E \) is defined as the energy of the electron in the electron-ion system. In the treatment of the excited states, the concept of the electron-ion system is used to determine the energy levels of the excited states. The energy levels of the hydrogen atom are determined by solving the Schrödinger equation for the electron in the Coulomb field of the nucleus. The function \( \phi(r) \) represents the motion of the electron in the Coulomb field.

The function \( \phi \) is a constant vector of arbitrary direction; the magnitude of \( \phi \) is

\[
\phi = g \left( \frac{2M + 1}{M} \right) = |\phi|
\]

where \( g \) is a constant vector of arbitrary direction, the magnitude of \( g \) is

\[
g = g \left( \frac{2M + 1}{M} \right) = |g|
\]

The equation for the function \( \phi \) is then the solution of the Schrödinger equation for the electron in the Coulomb field of the nucleus.

The usual procedure is to find the functions \( \phi \) in the form of a product function

\[
\phi = \phi(r)\phi(\theta, \phi)
\]

By the usual procedure, one finds that the functions \( \phi \) and satisfy the

\[
\left( \text{Eq. (L-7)} \right) \theta A \beta = \theta a \beta - \theta c \Delta \frac{\varepsilon}{\varepsilon} - \theta c \Delta \left( \frac{2M + 1}{M} \right)
\]

\[
\left( \text{Eq. (L-7)} \right) \theta A \beta = \theta a \beta - \theta c \Delta \frac{\varepsilon}{\varepsilon} - \theta c \Delta \left( \frac{2M + 1}{M} \right)
\]

By these substitutions, Eq. (L-7) becomes

\[
\left( \text{Eq. (L-7)} \right) \theta A \beta = \theta a \beta - \theta c \Delta \frac{\varepsilon}{\varepsilon} - \theta c \Delta \left( \frac{2M + 1}{M} \right)
\]

The hydrogen atom

\[
\left( \text{Eq. (L-7)} \right) \theta A \beta = \theta a \beta - \theta c \Delta \frac{\varepsilon}{\varepsilon} - \theta c \Delta \left( \frac{2M + 1}{M} \right)
\]

The hydrogen atom

\[
\left( \text{Eq. (L-7)} \right) \theta A \beta = \theta a \beta - \theta c \Delta \frac{\varepsilon}{\varepsilon} - \theta c \Delta \left( \frac{2M + 1}{M} \right)
\]

The hydrogen atom
\[ 0 = \begin{pmatrix} 1 + \frac{d}{(1 + 1)l} - \frac{d}{2} \end{pmatrix} + m \mathcal{D} - \nabla^2 - \mu_n \]

The differential equation for \( m \) is

\[ (\mu \mathcal{D} - \nabla^2) \exp m = n \]

where the solutions \( n \) are led to make

\[ \left[ \frac{d}{\eta_p} = \mu_n, \frac{d}{\eta_p} = n \right] \quad 0 = n(\mathcal{D} - \mu_n) \]

If \( \mu_n \) is large, \( \mathcal{L} \) is approximated by

\[ \frac{d}{\eta_p} = \mu_n \]

The boundary condition on the solutions of \( \mathcal{D} \) and are

\[ \mathcal{D} \exp \mu_n = \mu_n \]
\[ \omega'' - 2\sqrt{E} \omega' + \left( \frac{2}{r} - \frac{p (p+1)}{r^2} \right) \omega = 0 \]

\[ \omega = \sum_{k=0}^{\infty} a_k r^k \]

\[ \omega' = \sum_{k=0}^{\infty} k a_k r^{k-1} \quad \omega'' = \sum_{k=0}^{\infty} k(k-1) a_k r^{k-2} \]

\[ \sum_{k=0}^{\infty} \left\{ k(k+1) a_k r^{k-2} - 2\sqrt{E} k a_k r^{k-1} + \left( \frac{2}{r} - \frac{p (p+1)}{r^2} \right) a_k r^k \right\} = 0 \]

\[ \sum_{k=0}^{\infty} \left\{ \left[ k(k+1) - (p+1) k \right] a_k r^{k-1} + (-2 k\sqrt{E} + 2) a_k r^{k-1} \right\} = 0 \]

\[ \Rightarrow \quad k(k+1) - (p+1) k a_{k+1} + (-2 k\sqrt{E} + 2) a_k r^{k-1} = 0 \]

\[ a_{k+1} = 2 \frac{k\sqrt{E} - 1}{k(k+1) - (p+1) k} a_k \]

For \( r \to 0 \), \( \omega'' - 2\sqrt{E} \omega' \approx 0 \)

\[ \omega = e^{2\sqrt{E} r} \sqrt{E} (2\sqrt{E} - 1) e^{2\sqrt{E} r} = 0 \]

So, break off series at some value of \( k \), call it \( n \)

\[ F = 1/n^2 \quad a_{n+1} = \frac{2 n\sqrt{E} n^2 - 1}{a_n} = 0 \quad \text{so} \quad \omega = \sum_{k=0}^{n} a_k r^k \]
\[ \omega = \sum_{k=1}^{n} a_k r^k \]

\[ \begin{align*}
\begin{array}{c|c}
 n & l = 0, 1, 2, \ldots, n-1 \\
1 & 0 \\
2 & 0, 1 \\
3 & 0, 1, 2 \\
\end{array}
\end{align*} \]

That is, \(0 \leq l \leq n-1\), \(l = 0, 1, \ldots, n-1\).

Recall \(-l \leq m \leq l\).

For \(n = 1\), we have \(l = 0\), \(m = 0\) so nondegenerate.

For \(n = 2, \ldots\) degenerate.

For given \(l\), \((2l+1)\)-fold degeneracy from \(m\).

For given \(n\),

\[ S = \sum_{l=0}^{(2l+1)} = 2 \sum_{l=0}^{n-1} l + \sum_{l=0}^{n-1} 1 = 2 \sum_{l=1}^{n-1} l + n \]

But \[ \sum_{k=1}^{n} k = \frac{1}{2} n(n+1) \]

\[ S = 2\sum_{l=0}^{(2l+1)} + n = n^2 \]
\[0 = n \left[ \frac{\varepsilon_i^2}{(1 + l)l} \frac{1}{\mu} + \frac{1}{\varepsilon} + \frac{1}{\lambda} \right] + \frac{\varepsilon_i^2}{n_i^2} \]

The differential equation in \( L \) for the radial function \( n \) is

\[\frac{\partial}{\partial r} \left( r^2 \frac{\partial n}{\partial r} \right) + \frac{\varepsilon_i^2}{n_i^2} = 0 \]

The functions \( n \) are shown in Fig. 6-6.\(^{1} \)

\[c_i = \lambda_n \quad \varphi_i = \lambda_n \quad \varphi_i(\varepsilon) = \lambda_n \quad \lambda = \lambda_n \quad \lambda = \lambda_n \]

The functions \( n \) for \( l = 1, 2, 3, \ldots \) are

\[\psi_i = \psi_i(\varepsilon) \quad \lambda = \lambda_n \]

where the normalization constant \( \lambda \) is determined by

\[\langle 0 | 1 | n \rangle = 1 \quad \lambda \quad \lambda \quad \lambda \]

The orthogonal wave functions for the bound states of hydrogen are

\[\psi_i = \psi_i(\varepsilon) \quad \lambda = \lambda_n \]

where we have

\[\lambda = \lambda_n \]

Combining Eqs. (161-2)\(^{1} \), (169-2)\(^{1} \), and using Eq. (96-1), we obtain

\[\frac{\partial}{\partial r} \left( r^2 \frac{\partial n}{\partial r} \right) + \frac{\varepsilon_i^2}{n_i^2} = 0 \]

Multiplying by \( \varepsilon_i^2 \) and integrating, we obtain

\[\frac{\varepsilon_i^2}{n_i^2} (1 + \varepsilon) + \frac{1}{\mu} (1 + \varepsilon) = \frac{1}{n_i^2} \int (1 + \varepsilon) \psi_i^2 \left( \frac{1}{n_i^2} \right) \]

\[\lambda = \lambda_n \]

However, by the recurrence formula, we have

\[\lambda = \lambda_n \]

This normalization integral differs from Eq. (161-2)\(^{1} \), (169-2)\(^{1} \), and (170-2)\(^{1} \) by the factor \( \frac{1}{n_i^2} \), and the integral \( \int \frac{\psi_i^2}{n_i^2} \frac{1}{n_i^2} \) differs from the recurrence relation, the factor, rather than \( \lambda \), in that the integral

\[\lambda = \lambda_n \]

THE HYDROGEN ATOM
\[
\frac{\hbar}{\varepsilon - \left( \frac{\varepsilon^2}{2a} + \frac{1}{a} \right)^{\frac{1}{\gamma}}} = \mathcal{E}
\]

These points correspond closely to the extrema of the potential. The solutions of the energy equation (Eq. 1-13) are:

\[
L \left( \frac{1}{I} + \frac{1}{n} \right) - \frac{\varepsilon^2}{2I} + \frac{\varepsilon}{I} = \mathcal{E}
\]

\[
0 = \frac{\varepsilon^2}{(1 + \eta)I} - \frac{\varepsilon}{I} = \mathcal{E} \quad \text{and} \quad \mathcal{E} = \mathcal{E}
\]

The breathing points are the values \( \mathcal{E} \) for which

\[
(1 + \eta)I - \frac{\varepsilon^2}{2I} = \mathcal{E}
\]

The graph shows the potential for the hydrogen atom, \( \frac{\varepsilon^2}{I} \), \( \mathcal{E} \), in Fig. 7.2. The radial probability distribution function \( n^2 \) for several values

![Graph showing the radial probability distribution function](image-url)
(10-207) \( \langle A \rangle' = \frac{\hat{A}}{I} = \frac{\hat{A}}{I} + \hat{B} = \langle \hat{A} \rangle' + \langle \hat{B} \rangle = (A', B') = \langle \langle \hat{A} \rangle' \hat{B} \rangle' \)

In this equation, we are considering the expectation value of the Hamiltonian \( H \) with respect to the initial state \( \psi \). The expectation value is given by the inner product of the state with the Hamiltonian operator and the state itself.

(10-208) \( \frac{\epsilon}{2} (\epsilon^2 - 1) \frac{\phi_{\omega} v}{\epsilon^2} \) = \( d \) and \( \frac{\phi_{\omega} v}{\epsilon^2} = 0 \) \( v \)

The quantities \( \phi_{\omega} v \) have been substituted. The quantities \( d \) and \( v \)

(10-209) \( \phi_{\omega} v \) = \( \phi_{\omega} v \)

In the limit where the interaction is negligible, the phase \( \phi_{\omega} v \) drops out of the expression.

(10-207) \( \frac{0}{\epsilon} \int \frac{\phi_{\omega} v}{\epsilon} \) \( d \) \( w \) \( = \) \( \int \frac{d}{d} \) \( \int I = \frac{1}{I} \)

The time average of the quantity in the Bohr-Sommerfeld theory is

(10-207) \( \frac{\epsilon}{2} \int \langle \frac{\phi_{\omega} v}{\epsilon^2} \rangle \) \( 1 + \epsilon^2 \) \( \int \frac{\phi_{\omega} v}{\epsilon^2} \) \( = \frac{d}{d} \frac{1}{d} \) \( \frac{1}{d} \) \( \int I = \frac{1}{I} \)

For example, the Bohr-Sommerfeld quantization condition is given by

\[ \frac{\epsilon}{2} (\epsilon^2 - 1) \frac{\phi_{\omega} v}{\epsilon^2} = \frac{d}{d} \frac{1}{d} \int I = \frac{1}{I} \]

References

Examples of this kind are discussed in Problem 7-40.

Further

References
The second term of the solution (6.2) leads, at \( r = 0 \), to 

\[
\int_{0}^{\frac{\pi}{2}} d\xi = \frac{\pi}{2}
\]

and the norm is the normalization integral.

\[
1 = \frac{[f]}{\int_{0}^{2\pi} d\phi} = \frac{[f]}{\int_{0}^{2\pi} d\phi}
\]

The energy integral \( E \approx \frac{1}{2} \) for \( r = 0 \) and small \( r \), this solution would become the kinetic

\[
\frac{1}{2} \left( 1 - \frac{r^2}{\xi} \right) + \frac{z}{\cos \xi} = \frac{1}{2} \left( 1 - \frac{r^2}{\xi} \right) + \frac{z}{\cos \xi} = (z)^{1/2}
\]

\[
\frac{y}{\sin \xi} = (z)^{1/2}
\]

For the lower \( l \) values, the spherical Bessel functions are zeros.

For each \( l \) with \( n = l + 1 \), the radial quantum number counting the levels number of zeros, we find an infinite number of values of \( l \) and of energy.

Since for each given value of \( l \), the Bessel function has an infinite number of zeros, we find an infinite number of values of \( l \) and of energy.

From this set of solutions we select the eigenfunctions by the condition

\[
\langle \mu, \gamma \rangle | \psi \rangle = (j) \chi
\]

are restricted to be identical at the origin (cf. also Problem 6.1). Hence, normalized solutions

\[
\int_{0}^{2\pi} d\phi = \frac{\pi}{2}
\]

\[
E = \frac{1}{2} \left( 1 - \frac{r^2}{\xi} \right) + \frac{z}{\cos \xi} = \frac{1}{2} \left( 1 - \frac{r^2}{\xi} \right) + \frac{z}{\cos \xi} = (z)^{1/2}
\]

\[
\frac{y}{\sin \xi} = (z)^{1/2}
\]
\[ 0 = \chi \left( \frac{r^d}{(1 + l)^l} - I^2 \right) + \chi \]

and

\[ 0 = \chi \left( \frac{r^d}{(1 + l)^l} - I^2 \right) + \chi \]

the radial part of the Schrödinger equation becomes

\[ r^d = \frac{\hbar^2}{2m} \frac{2^l (l!)^2}{l^l} \]

Solution with abbreviations

size \( W_0 \): \( P_0 \)

For \( l = 0 \), \( 1/2 \), \( 1 \), \( 3/2 \), \( 2 \), \( 5/2 \), \( 3 \), \( 7/2 \), \( 4 \), \( 9/2 \), \( 5 \), \( 11/2 \), \( 6 \), \( 13/2 \), \( 7 \), \( 15/2 \), \( 8 \), \( 17/2 \), \( 9 \), \( 19/2 \), \( 10 \), \( 21/2 \), \( 11 \), \( 23/2 \), \( 12 \), \( 25/2 \), \( 13 \), \( 27/2 \), \( 14 \), \( 29/2 \), \( 15 \), \( 31/2 \), \( 16 \), \( 33/2 \), \( 17 \), \( 35/2 \), \( 18 \), \( 37/2 \), \( 19 \), \( 39/2 \), \( 20 \)

To determine the eigenvalues of bound states in the square well potential.

Problem 63. Square well of finite depth.

Table of the values of \( x \) for

\( x = \frac{\hbar^2}{2m} \frac{2^l (l!)^2}{l^l} \)

<table>
<thead>
<tr>
<th>l</th>
<th>( x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.316</td>
</tr>
<tr>
<td>1</td>
<td>1.721</td>
</tr>
<tr>
<td>2</td>
<td>1.159</td>
</tr>
<tr>
<td>3</td>
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</tr>
<tr>
<td>4</td>
<td>0.450</td>
</tr>
<tr>
<td>5</td>
<td>0.244</td>
</tr>
<tr>
<td>6</td>
<td>0.136</td>
</tr>
<tr>
<td>7</td>
<td>0.081</td>
</tr>
<tr>
<td>8</td>
<td>0.045</td>
</tr>
<tr>
<td>9</td>
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<td>10</td>
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</tr>
<tr>
<td>11</td>
<td>0.007</td>
</tr>
<tr>
<td>12</td>
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</tr>
<tr>
<td>13</td>
<td>0.002</td>
</tr>
<tr>
<td>14</td>
<td>0.001</td>
</tr>
<tr>
<td>15</td>
<td>0.001</td>
</tr>
<tr>
<td>16</td>
<td>0.001</td>
</tr>
</tbody>
</table>

Recurrence relation

\[ (z)^{1 - l - (z)^{-1} - l^{-1} - z} = (z)^{l} \]

and for higher values of \( l \) they may easily be constructed from the

\[ \{ \text{etc} \} \]

The lowest energy values have been expressed in multiples of \( \hbar^2/2m \).

They all tend towards either \( m \) for even and \( m + 1/2 \) for odd.
functions:

We also note the relation between the spherical and the ordinary Bessel functions:

\[ f(\rho) \sim \cos(\rho) \quad (\rho \ll |\rho| \quad \text{and} \quad 1 \ll |\rho|) \]

This expression, further integrated by parts, and the leading term is the asymptotic form of the spherical Bessel functions can be derived from

\[ sp(\rho) \delta_{1+\rho} \int_{1+\rho}^{\infty} \frac{d\xi}{\xi} = (x)'f \]

where the definition (9.67) of the total angular momentum:

\[ A(\rho) = (\rho) \]


We thus see that the spherical Bessel function (x)'f is the solution of the radial equation:

\[ (x+\rho)O + \frac{1}{1+\rho} \frac{1}{i(1+\rho)} = (x)'f \]

Powers of \( x \) are generally not considered, but here we are interested in the behavior of the function \( f(x) \) as \( x \to \infty \). Therefore, we have

\[ sp(\rho) = \int_{1+\rho}^{\infty} \frac{d\xi}{\xi} \]

This solution is closely related to the spherical Bessel function, defined by

\[ \delta_{1+\rho} \int_{1+\rho}^{\infty} \frac{d\xi}{\xi} = (x)'f \]

This equation was introduced in detail in Section 7.5. The solution of

\[ 0 = \frac{\gamma d}{1+\rho} - 1 + \frac{\delta d}{\delta d} \]

and

If we substitute

\[ \delta(\rho) = \delta(\rho) \]

The differential equation (9.67) can be solved by standard power series

\[ \left( \frac{\gamma d}{1+\rho} \right) \left( \frac{\delta d}{\delta d} \right) = (x)'f \]

We now consider the eigenvalues in the form (9.72) of the Schrödinger equation for the free particle, which is given at the origin and is said to be regular. The radial equation:

\[ \Delta = \rho \left( \frac{1+\rho}{1-\rho} \right) \frac{1}{p} - \]

which has solutions the form of the solution of the form (9.72) of the Schrödinger equation. The eigenvalues must be a constant of the motion, and the angular momentum is a constant of the motion. The eigenvalues of the problem are then the eigenvalues of the radial Schrödinger equation. Since the radial Schrödinger equation is of the form (9.72) of the Schrödinger equation, the problem cannot be solved by standard power series methods. We thus consider the eigenvalues in the form (9.72) of the Schrödinger equation.

\[ 0 = \frac{\gamma d}{1+\rho} - 1 + \frac{\delta d}{\delta d} \]

and for \( \rho \to 0 \), the central equation:

\[ \delta(\rho) \Delta A(\rho) = \delta(\rho) \Delta A(\rho) \]


Chapter 10: Constrained Potentials and the Hydrogen Atom

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The spherical harmonic expansion of the plane waves.

The spherical harmonic expansion of plane waves.

\[
\frac{d}{z_\nu(1+i) - d} \sin (d'y) = (z)_{(0)} y
\]

For \( z \approx 0 \),

\[
(z)_{(0)} y - (z)_{(1)} y = (z)_{(0)} y
\]

and

\[
(z)_{(0)} y + (z)_{(1)} y = (z)_{(1)} y
\]

Hence,

\[
\frac{[z(0) y - z(1) y]}{1} = (z)_{(1)} y
\]

It is useful to define also the spherical Neumann function as

\[
(z)_{(1)} y + (z)_{(1)} y = (z)_{(1)} y
\]

The explicit forms of the spherical Bessel, Hankel, and Neumann functions.

For \( \nu = 0, 1, \ldots \), the given below are the explicit forms of the spherical Bessel, Hankel, and Neumann functions.

Chapter 10: Comperm Ptotentials and the Hydrogen Atom
Fig. 7-6. The radial probability distribution function $\psi^2_{n,l}$ for several values of the quantum numbers $n, l$. (From E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra*, Cambridge University Press, Cambridge, 1953, by permission.)
Fig. 35. Energy levels for a particle enclosed in a sphere
<table>
<thead>
<tr>
<th></th>
<th>Table of the values of $x^n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>
The kinetic energy operator in 3D

Classically, \( K = \frac{1}{2m} \mathbf{p}^2 + \frac{1}{2k} r^2 \mathbf{L}^2 \)

\( \mathbf{p}^r \) is radial component of the momentum.

What is \( \mathbf{p}^r \)?

\[
(A \times B) \cdot (C \times D) = (A \cdot C)(B \cdot D) - (A \cdot D)(B \cdot C)
\]

(just write out in Cartesian components)

If we want to deal with \( \mathbf{p}^r \) need to keep order straight.

\[
[i \mathbf{r}_\alpha \mathbf{p}_\beta] = i \mathbf{r}_\alpha \mathbf{p}_\beta - i \mathbf{r}_\beta \mathbf{p}_\alpha
\]

\[
\mathbf{r} \mathbf{p} = \sum_{\alpha \beta} A_\alpha B_\beta \mathbf{D}_\beta - \sum_{\alpha \beta} A_\alpha D_\beta B_\beta \mathbf{C}_\beta & \text{ dot products.}
\]

\[
= \sum_{\alpha \beta} \left[ A_\alpha B_\beta C_\alpha D_\beta - A_\alpha B_\beta C_\beta D_\alpha \right] & \text{ keep order over.}
\]

\[
\mathbf{L}^2 = \mathbf{L} \cdot \mathbf{L} = (\mathbf{r} \times \mathbf{p}) \cdot (\mathbf{r} \times \mathbf{p})
\]

\[
= \sum_{\alpha \beta} \left[ \mathbf{r}_\alpha \mathbf{p}_\beta \mathbf{r}_\beta \mathbf{p}_\alpha - \mathbf{r}_\alpha \mathbf{p}_\beta \mathbf{r}_\beta \mathbf{p}_\alpha \right]
\]

\[
= \sum_{\alpha \beta} \left[ \left( i \mathbf{r}_\alpha \mathbf{p}_\beta \mathbf{r}_\beta \mathbf{p}_\alpha - i \mathbf{r}_\beta \mathbf{p}_\beta \mathbf{r}_\beta \mathbf{p}_\alpha \right) - \left( i \mathbf{r}_\beta \mathbf{p}_\beta \mathbf{r}_\beta \mathbf{p}_\alpha + i \mathbf{r}_\beta \mathbf{p}_\beta \mathbf{r}_\beta \mathbf{p}_\alpha \right) \right]
\]

\[
= r^2 \mathbf{p}^2 - 2i \mathbf{r} \mathbf{p} - \sum_{\alpha \beta} \left( i \mathbf{r}_\beta \mathbf{p}_\beta \mathbf{r}_\beta \mathbf{p}_\alpha - i \mathbf{r}_\beta \mathbf{p}_\beta \mathbf{r}_\beta \mathbf{p}_\alpha \right)
\]

\[
= r^2 \mathbf{p}^2 - 2i \mathbf{r} \mathbf{p} - \left( i \mathbf{r} \mathbf{p} \right) \left( \mathbf{r} \cdot \mathbf{p} \right) + 3 r \cdot \mathbf{p}
\]

\[
\mathbf{L}^2 = r^2 \mathbf{p}^2 + i \mathbf{r} \mathbf{p} - \left( \mathbf{r} \cdot \mathbf{p} \right) \left( \mathbf{r} \cdot \mathbf{p} \right)
\]
Solve for $p^2$

$$r^2p^2 = L^2 - i\hbar (\mathbf{r} \cdot \mathbf{p}) + (\mathbf{r} \cdot \mathbf{p})^2$$

$$p^2 = \frac{1}{r^2} (\mathbf{r} \cdot \mathbf{p})^2 - \frac{i\hbar}{r^2} \mathbf{r} \cdot \mathbf{p} + \frac{L^2}{r^2}$$

$$(\mathbf{r} \cdot \mathbf{p})^2$$

Define a new operator $\mathbf{p}_r$ - the radial component of the momentum -

$$\mathbf{p}_r = \frac{1}{r} \mathbf{r} \times \mathbf{p} + \mathbf{r} \times \frac{\mathbf{p}}{r} = \frac{1}{r} (\mathbf{r} \times \mathbf{p} + \mathbf{p} \times \mathbf{r})$$

$$\hat{\mathbf{r}} \cdot \mathbf{r} = r \cdot \mathbf{r} = \frac{r^2}{r} = 1.$$

Comparing this with the classical version of $\mathbf{p}_r$,

$$\mathbf{p}_r = \hat{\mathbf{r}} \cdot \mathbf{p}$$

i.e. $\mathbf{p}$ takes the radial component of the momentum.

If we tried to use $\mathbf{r} \cdot \mathbf{p}$ as operator in QM it is not Hermitian. But, you know that $AB + BA$ is Hermitian, even if $AB$ is not, $(AB)^+ = B^+ A^+$

$$(AB + BA)^+ = (AB)^+ + (BA)^+ = B^+ A^+ + A^+ B^+ = B^+ A^+ + B^+ A^+$$

when $A + B$ are Hermitian. $= AB + BA$ Q.E.D.
You show that \[ [\mathbf{r}, \mathbf{p}] = i \hbar \] where \( \mathbf{p} = \hat{\mathbf{r}} \cdot \mathbf{p} \)

First show that \[ [\mathbf{r}, \mathbf{p}] = i \hbar \] then \( \hat{\mathbf{r}} \cdot [\mathbf{r}, \mathbf{p}] = \mathbf{r} \hat{\mathbf{p}} - \hat{\mathbf{p}} \mathbf{r} = \mathbf{r} \cdot \mathbf{p} - \mathbf{p} \cdot \mathbf{r} = \hbar \)

Want to write \( \mathbf{p} \cdot \mathbf{p} \) in terms of \( \mathbf{p} \).

Remember \( [\mathbf{p}, f] = \frac{\hbar}{i} \frac{\partial f}{\partial \mathbf{r}} \)

\[
[\mathbf{p}, f] = \frac{\hbar}{i} \left( \frac{\partial f}{\partial \mathbf{r}} \cdot \hat{\mathbf{r}} + \frac{\partial f}{\partial \mathbf{r}} \cdot \mathbf{e} \right) = \frac{\hbar}{i} \mathbf{r} \cdot \mathbf{e} = \frac{\hbar}{i} \mathbf{r} \cdot \mathbf{e}
\]

\[ \mathbf{p} = \frac{\hbar}{i} \left( \frac{\hbar}{i} \mathbf{r} \cdot \mathbf{e} + \mathbf{e} \cdot \frac{\hbar}{i} \right) = \frac{\hbar}{i} \left( \frac{\hbar}{i} \mathbf{r} \cdot \mathbf{e} + \frac{\hbar}{i} \mathbf{e} \cdot \mathbf{r} + \frac{\hbar}{i} \mathbf{e} \cdot \mathbf{e} \right) \]

\[ p_{\mathbf{r}} = \frac{\hbar}{i} \left( \frac{\hbar}{i} \mathbf{r} \cdot \mathbf{e} - \frac{\hbar}{i} \mathbf{e} \cdot \mathbf{r} \right) \]

\[ \mathbf{p} \cdot \mathbf{p} = \mathbf{p}_{\mathbf{r}} + \mathbf{r} \mathbf{p} + i \hbar \]

\[ \mathbf{p} \cdot \mathbf{p} = \mathbf{p}_{\mathbf{r}} + \mathbf{r} \mathbf{p} + i \hbar \]

Then (see pg 2 X)

\[ \frac{1}{r^2} (\mathbf{r} \cdot \mathbf{p})^2 - \frac{\hbar^2}{r^2} (\mathbf{r} \cdot \mathbf{p}) = \frac{1}{r^2} \left( \mathbf{r} \cdot \mathbf{p} - i \hbar \right) (\mathbf{r} \cdot \mathbf{p}) \]

\[ = \frac{1}{r^2} \left( \mathbf{p}_{\mathbf{r}} + i \hbar \mathbf{p}_{\mathbf{r}} \right) (\mathbf{r} \cdot \mathbf{p}) = \frac{1}{r} \mathbf{p}_{\mathbf{r}} (\mathbf{r} \cdot \mathbf{p}) = \frac{1}{r} \mathbf{p}_{\mathbf{r}} (\mathbf{r} \cdot \mathbf{p}) \]

\[ = \frac{1}{r} \left( \mathbf{p}_{\mathbf{r}} \mathbf{p}_{\mathbf{r}} + i \hbar \mathbf{p}_{\mathbf{r}} \right) = \frac{1}{r} \left( \mathbf{p}_{\mathbf{r}} \mathbf{p}_{\mathbf{r}} + i \hbar \mathbf{p}_{\mathbf{r}} + i \hbar \mathbf{p}_{\mathbf{r}} \right) = \mathbf{p}_{\mathbf{r}}^2 \]

\[ \mathbf{p}^2 = \mathbf{p}_{\mathbf{r}}^2 + \frac{\hbar^2}{r^2} \quad K = \frac{\hbar^2}{2m} \mathbf{p}_{\mathbf{r}}^2 + \frac{i}{2m} \mathbf{p}_{\mathbf{r}}^2 \]

From \( \mathbf{r} \mathbf{p} = \mathbf{p} \mathbf{r} = i \hbar \)
Return to the S equation. Let us set $V=0$ to do the 3D free particle problem.

$$H = \frac{1}{2m} (p_r^2 + p_\theta^2 + p_\phi^2)$$

$$= \frac{1}{2m} p_r^2 + \frac{1}{2m} r^2 L^2$$

Look for eigenvalues of $H, L_z, L^2$.

Useful for theory of scattering.

Incident plane wave $k \vec{E} \rightarrow \text{scattered wave } k \vec{E} \rightarrow \text{spherical}$.

Far from material the waves are free particle (3D).

The $S$ wave is

$$\left[ \frac{1}{2m} p_r^2 + \frac{1}{2} r (k+1) \frac{k^2}{r^2} \right] Y_{klm}(\theta \phi) = \frac{\hbar^2 k^2}{2m} \left( \frac{k}{r} \right) Y_{klm}(\theta \phi)$$

where $E = \hbar^2 k^2 / 2m$ is free particle energy.

The radial momenta operator is

$$p_r = \frac{i}{\hbar} r \cdot \vec{p} - \frac{i \hbar}{\hbar}$$

Now,

$$\vec{p} = \frac{\hbar}{i} \left( \frac{d}{dr} \hat{r} + \hat{r} \frac{d}{dr} \right) \text{ in sph. coord., so}$$

$$r \cdot \vec{p} = \frac{\hbar}{i} r \frac{d}{dr} \hat{r} = r \frac{\hbar}{i} \frac{d}{dr} \hat{r} = r \frac{\hbar}{i} \frac{d}{dr}$$

$$p_r = \frac{\hbar}{i} \left( \frac{d}{dr} + \frac{1}{r} \right).$$
And, we may write this as

\[ \rho_r = \frac{\hbar}{ir} \frac{d}{dr} \left( r \frac{df}{dr} + f \right) = \frac{\hbar}{i} \left( r \frac{d}{dr} + \frac{1}{r} \right)f \]

\[ \rho_r \rho_r f = -\hbar^2 \frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} f \right) = -\hbar^2 \frac{1}{r} \frac{d^2}{dr^2} rf \]

Then \( S \) as is for radial part

\[ \int \left[ -\frac{\hbar^2}{2m} \frac{1}{r^2} \frac{d^2}{dr^2} r + \frac{\hbar^2}{2mr^2} \delta(k^2) \right] R(k) = E \Pi(k) \]

This diff. eq. can be solved by standard power series solution method (Essentially same method as used for Legendre).

Another way: set \( \rho = \sqrt{\frac{2mE}{\hbar^2}} r = kr \) \( \rho(k) = r R(r) \)

and \( \lambda = k^2 + 1 \) or \( \lambda = -k^2 \) (Remake Legendre)

Then diff. eq. becomes

\[ \frac{d^2 v}{dr^2} + \left[ 1 - \frac{\lambda(\lambda-1)}{\rho^2} \right] v = 0 \]

* \[ \frac{1}{r} \frac{d^2}{dr^2} f = \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{df}{dr} \right) \]

\[ \text{both} = \frac{2f'}{r} + f'' \]
The solution of this equation defines, for the
solution that vanishes at $r = 0$, the
spherical Bessel function of the first kind. (Regular sol.)

$$R(r) = C j_0(kr)$$

They are related to the ordinary Bessel functions as

$$j_0(z) = \sqrt{\frac{\pi}{2z}} J_{1/2}(z)$$

$$\begin{align*}
j_0(kr) &= \frac{\sin kr}{kr} \\
j_1(kr) &= \frac{\sin kr}{(kr)^2} - \frac{\cos kr}{kr}
\end{align*}$$

See p.6.
Figure 8.6: Plots of the Spherical Bessel Functions $j_0(x)$ and $j_1(x)$. 