Problem Set 4
(Due 11/02)

1. Show that \((\Delta p)^2 \equiv \langle \psi(t) | \hat{p}^2 | \psi(t) \rangle - \langle \langle \psi(t) | \hat{p} | \psi(t) \rangle \rangle^2\) for a free particle satisfies
\[
\frac{d}{dt}(\Delta p)^2 = 0.
\]

2. Prove the quantum mechanical virial (virial is Latin for force) theorem:
\[
\left\langle \psi(t) \left| \hat{x} \frac{\partial V}{\partial \hat{x}} \right| \psi(t) \right\rangle = 2 \left\langle \psi(t) | \hat{K} | \psi(t) \right\rangle,
\]
where \(V\) is the potential energy and \(\hat{K}\) the kinetic energy operator, and the wavefunction corresponds to a state of definite energy.
Hint: Consider the commutator, \([\hat{H}, \hat{x} \hat{p}]\).

3. Show that the matrix elements \(x_{kn}\) and \(p_{kn}\) of the position and momentum operators for stationary states (thus using wavefunctions \(|\psi_k\rangle\) for an energy representation) satisfy
\[
\frac{E_n - E_k}{i\hbar} x_{kn} = \frac{1}{m} p_{kn}.
\]

4. Consider the matrix \(H = \begin{pmatrix} E_0 & -A \\ -A & E_0 \end{pmatrix}\). Diagonalize it by finding its eigenvalues and eigenfunctions using matrix methods. That means first obtain the eigenvalues from the “secular determinant” \(H - \lambda I\) where \(I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\). Then substitute each of the two eigenvalues in turn into the secular equation and solve for each eigenfunction.
Note: each eigenfunction is a vector with two elements.
How does this compare with what was found in the notes on the two level system?

5. Using the raising and lowering operators show that the harmonic oscillator uncertainty relation \((\Delta x)(\Delta p) = \frac{\hbar}{2}\) where \((\Delta p)\) is defined in problem 1 and there is an analogous definition for \((\Delta x)\) holds for the ground state. You should first work out \((\Delta x)\) and \((\Delta p)\) for any (oscillator) state.

6. At \(t=0\) a particle in a harmonic oscillator potential is in the initial state:
\[
|\psi(0)\rangle = \frac{1}{\sqrt{5}} |E_1\rangle + \frac{2}{\sqrt{5}} |E_2\rangle.
\]
(a) What is the expectation value of the energy in this state?
(b) Find \(|\psi(t)\rangle\). Is it a stationary state?
(c) Evaluate the expectation value $\langle \psi(t) | \hat{x} | \psi(t) \rangle$. What is the frequency of oscillation of this expectation value?

7. Show that if $| E_n \rangle$ is the nth excited state of a harmonic oscillator then

$$\langle E_n | x | E_{n+1} \rangle = \sqrt{\frac{n+1}{2}} x_0$$

$$\langle E_n | x^2 | E_{n+1} \rangle = 0$$

$$\langle E_n | x | E_{n+2} \rangle = 0$$

$$\langle E_n | x^2 | E_{n+2} \rangle = \frac{1}{2} \sqrt{n+2} \sqrt{n+1} x_0^2$$