1. a) Show that \((AB)^+ = B^+A^+\) by using the definition of adjoint operator given in the notes.
b) Suppose you have two Hermitian operators. Do they necessarily commute?
c) Suppose the two operators do commute. Is their product Hermitian.

(Note: a) does generalize to \((ABC)^+ = (C^+B^+A^+)\) etc.)
(Note: a) is an important property – used all the time).

2. Using the results of 1, consider the following operator products:
\(\hat{p}\hat{x}, \hat{p}\hat{x} + \hat{x}\hat{p}, \hat{x}^2\hat{p}, \hat{x}^2\hat{p}\) and \(i\left(\hat{x}^2\hat{p} - \hat{p}\hat{x}^2\right)\).
Which are Hermitian? Show how you arrive at your conclusions.

3. Show that if \(A\) is a Hermitian operator, then the expectation value of \(A^2\) is non-negative (that is \(0 \leq \langle\psi|A^2|\psi\rangle\)). Also show that the all the eigenvalues of \(A^2\) are non-negative numbers.

4. Show that the kinetic energy operator is Hermitian. Show that the Hamiltonian operator is Hermitian. (You may assume that you already showed that the potential energy operator is Hermitian.)

5. Show that
\[
\]
and
\[
[A, BC] = [A, B]C + B[A, C]
\]

Note: You don’t have to do the following. But it is important to know that these relations can be used to show that
\[
[A, B^n] = nB^{n-1}[A, B]
\]
\[
[A^n, B] = nA^{n-1}[A, B]
\]

if \(A\) and \(B\) commute with their commutator \([A, B]\). (Note that it is important to realize/justify that this statement also implies that \([A, B]\) will commute with any power of \(A\) and \(B\).)
6-7. Consider the energy eigenvectors/eigenvalues of the particle in an (infinite) square well with position representation eigenvectors $\psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n \pi x}{L}$.

In abstract notation, the eigenvectors are denoted as $|E_i\rangle$, $i=1,2,\ldots$. We will keep the “$E$” in the notation to make sure it is clear that one is working in this energy basis set.

Suppose at a given instant, the particle is in the superposition state:

$$|\psi\rangle = \frac{1}{2} |E_1\rangle - \frac{\sqrt{3}}{2} |E_3\rangle.$$  

a) Express this state in the $x$ representation.

b) Express this state in the $p$ (momentum) representation. (Hint: Use a Fourier transform on the $x$ representation. The Mathematica results below should be useful.)

c) Express this state in the energy representation. (Hint: in the energy representation the state is diagonal, since $\hat{H}\psi_n = E_n \psi_n$.)

d) Write down the energy operator $\hat{H} = \hat{p}^2/2m$ in each of these representations.

e) Calculate the expectation value of the energy. Do it three times using the three representations.

(You may have trouble with the momentum integration – just set it up so that the result is expressed in terms of an integral that yields a pure number. See below for Mathematica results.)

(Hint: For the energy representation use the results of problem set 2.3.)

f) Using whichever representation you think is most convenient, relying on the “a lazy scientist is a good scientist” idea, find the root mean square deviation of the energy from the mean energy.

Appendix for 3.6-7 b)

$$\text{Integrate}[\text{Exp}[-i p x] \cdot \text{Sin}[\text{Pi} \cdot x/L], \{x, 0, L\}]$$

$$\frac{(1 + e^{-1Lp}) L \pi}{12 L^2 p^2 + \pi^2}$$

$$\text{Integrate}[\text{Exp}[-i p x] \cdot \text{Sin}[3 \cdot \text{Pi} \cdot x/L], \{x, 0, L\}]$$

$$\frac{3 (1 + e^{-1Lp}) L \pi}{12 L^2 p^2 + 9 \pi^2}$$
Appendix for 3.6-7 e)

\[ \text{Integrate}\left[ z^2(1+\cos[z\pi])/(1-z^2)^2,\{z,0,\text{Infinity}\}\right], \text{PrincipalValue}\rightarrow \text{True} \]

\[ \pi^3 \text{BesselJ}\left[ \frac{3}{2}, \pi \right] \]

\[ 4\sqrt{2} \]

\[ \text{Integrate}\left[ z^2(1+\cos[z\pi])\left(1/(1-z^2) - 3\sqrt{3}/(9-z^2)\right)^2,\{z,0,\text{Infinity}\}\right], \text{PrincipalValue}\rightarrow \text{True} \]

\[ \frac{72}{108} \left( \frac{7i\pi}{512} + \frac{9\log[3]}{512} - \frac{9}{512} (i\pi + \log[3]) \right) - 60\sqrt{3} \left( \frac{3i\pi}{512} + \frac{7\log[3]}{512} - \frac{7}{512} (i\pi + \log[3]) \right) - \]

\[ \frac{28}{1536} \left( \frac{11i\pi}{512} - \frac{27\log[3]}{512} + \frac{27}{512} (i\pi + \log[3]) \right) + 6\sqrt{3} \left( \frac{11i\pi}{512} - \frac{27\log[3]}{512} + \frac{27}{512} (i\pi + \log[3]) \right) \]

\[ \text{Simplify}[\%] \]

\[ \frac{1}{4} i (-10 + 3\sqrt{3}) \pi \]

\[ \text{Integrate}\left[ z^2(1+\cos[z\pi])\left(3\sqrt{3}/(9-z^2)\right)^2,\{z,0,\text{Infinity}\}\right], \text{PrincipalValue}\rightarrow \text{True} \]

\[ \frac{9}{4}\sqrt{\frac{3}{2}} \pi^2 \left( -2 \text{BesselJ}\left[ \frac{1}{2}, 3\pi \right] + 3\pi \text{BesselJ}\left[ \frac{3}{2}, 3\pi \right] \right) \]

\[ \text{Integrate}\left[ z^2(1+\cos[z\pi])\left(2*3\sqrt{3}/(9-z^2)\right)\left(1/(1-z^2)\right),\{z,0,\text{Infinity}\}\right], \text{PrincipalValue}\rightarrow \text{True} \]

\[ -108\sqrt{2} \pi^2 \text{BesselJ}\left[ \frac{1}{2}, 3\pi \right] \]

\[ N[\text{BesselJ}\left[1/2,3\pi\right]] \]

\[ 2.87626 \times 10^{-17} \]

\[ N[\text{BesselJ}\left[3/2,\pi\right]] \]

\[ 0.450158 \]
\[ N[\text{BesselJ}[3/2,3\pi]] \]
\[ 0.259899 \]
\[ \text{BesselJ}[3/2,x]/.x\rightarrow\pi \]
\[ \frac{\sqrt{2}}{\pi} \]
\[ \text{BesselJ}[3/2,x]/.x\rightarrow3\pi \]
\[ \frac{\sqrt{\frac{2}{3}}}{\pi} \]
\[ \text{BesselJ}[1/2,x]/.x\rightarrow3\pi \]
\[ 0 \]

\[ \pi^3 1/(4\sqrt{2}) \cdot \sqrt{2}/\pi + \]
\[ 9/4 \cdot \sqrt{3/2} \cdot \pi^2 \cdot 3\pi \cdot \sqrt{2/3}/\pi \]
\[ 7\pi^2 \]