1. From the potential step solution on pages 89-90 of the notes (see Eq. 6.32) show that indeed

\[ \frac{B}{A} \frac{ik + \kappa}{ik - \kappa} = e^{\alpha} \text{ where } \alpha \text{ is a real number.} \]

If this is to be an identity, we first express the right hand side as

\[ e^{\alpha} = \cos \alpha + i \sin \alpha \]

and the left hand side as

\[ \frac{ik + \kappa}{ik - \kappa} \frac{ik + \kappa}{ik - \kappa} = \frac{k^2 - \kappa^2}{k^2 + \kappa^2} + i \left( \frac{-2k\kappa}{k^2 + \kappa^2} \right) \]

Then this means

\[ \cos \alpha = \frac{k^2 - \kappa^2}{k^2 + \kappa^2} \]

and

\[ \sin \alpha = \sqrt{1 - \cos^2 \alpha} = \sqrt{1 - \left( \frac{k^2 - \kappa^2}{k^2 + \kappa^2} \right)^2} = \sqrt{\frac{4k^2\kappa^2}{(k^2 + \kappa^2)^2}} = \frac{-2k\kappa}{k^2 + \kappa^2} \]

taking the negative root, which agrees with the \( e^{\alpha} = \cos \alpha + i \sin \alpha \) expression.

2. For the Schrödinger equation

\[ -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi(x) = E\psi(x) \]

with \( E < V \) (and since \( V \) is 0 for part of the range, this means \( E < 0 \)), it is convenient to define a positive constant \( k^2 \) via \( \frac{\hbar^2k^2}{2m} = V - E \).

Then,

\[ \frac{\partial^2 \psi(x)}{\partial x^2} - k^2 \psi(x) = 0 \]

with solutions
\[ \psi(x) = Ae^{kr} + Be^{-kr}, \]

where A and B are arbitrary complex constants. Using the boundary conditions of wavefunction continuity at \( \pm L/2 \), one gets

\[ Ae^{kL/2} + Be^{-kL/2} = 0 \]  \hspace{1cm} (1)

\[ Ae^{-kL/2} + Be^{kL/2} = 0 \]  \hspace{1cm} (2)

Multiply (1) by \( e^{kL/2} \) to get

\[ B = -Ae^{kL} \, \text{so} \]

\[ Ae^{-kL/2} - Ae^{kL} e^{kL/2} = 0 \text{ or} \]

\[ 1 - e^{2kL} = 0 \]

This says \( kL = 0 \). Since \( k \) must be nonzero by assumption, we have a contradiction. Therefore, there are no states with corresponding negative energy.

3. To show that

\[ \left\langle \left( \hat{p}^2 / 2m \right)^\ell \right\rangle = \sum |a_n|^2 \left( E_n \right)^\ell, \]

use the “peel off” method:

\[ \left\langle \left( \hat{p}^2 / 2m \right)^\ell \right\rangle = \int \psi^*(x,t) \left( \hat{p}^2 / 2m \right)^\ell \psi(x,t) \, dx \]

\[ = \int \left( \sum a_n \psi_n(x) \right)^* \left( \left( \hat{p}^2 / 2m \right)^\ell \right) \left( \sum a_m \psi_m(x) \right) \, dx \]

Then write

\[ \left( \hat{p}^2 / 2m \right)^\ell = \left( \hat{p}^2 / 2m \right)^{\ell-1} \left( \hat{p}^2 / 2m \right). \]

Since \( \left( \hat{p}^2 / 2m \right) \psi_m(x) = E_m \psi_m(x) \) and \( E_m \) is a number, we have

\[ \left( \hat{p}^2 / 2m \right)^{\ell-1} \left( \hat{p}^2 / 2m \right) \psi_m(x) = E_m \left( \hat{p}^2 / 2m \right)^{\ell-1} \psi_m(x). \]

Proceeding inductively, this yields
\[
\left( \frac{\hat{p}^2}{2m} \right)^\ell \psi_m(x) = (E_m)^\ell \psi_m(x),
\]
which then leads to the desired result.

4. The superposition has \( a_1(0) = \frac{\sqrt{3}}{2} \) and \( a_2(0) = \frac{1}{2} \); therefore, the probabilities are \( |a_1(0)|^2 = \frac{3}{4} \) and \( |a_2(0)|^2 = \frac{1}{4} \). The expectation of the energy

\[
\langle E_n \rangle = \sum_{n=1}^{\infty} |a_n|^2 E_n^p = \frac{3}{4} E_1^p + \frac{1}{4} E_2^p. \quad (p = 0, 1, \ldots)
\]

For \( E_n = n^2 \hbar^2 \pi^2 / 2mL^2 \) \( (n = 1, 2, \ldots) \)

\[
\langle E \rangle = \frac{\hbar^2 \pi^2}{2mL^2} \left[ \frac{3}{4} \cdot 1 + \frac{1}{4} \cdot 4 \right] = \frac{7 \pi^2 \hbar^2}{4 2mL^2}
\]

\[
\langle E^2 \rangle = \left( \frac{\hbar^2 \pi^2}{2mL^2} \right)^2 \left[ \frac{3}{4} \cdot 1^2 + \frac{1}{4} \cdot 4^2 \right] = \frac{19}{4} \left( \frac{\pi^2 \hbar^2}{2mL^2} \right)^2
\]

\[
\Delta E \equiv \left[ \langle E^2 \rangle - \langle E \rangle^2 \right]^{1/2} = \frac{3\sqrt{3}}{16} \left( \frac{\pi^2 \hbar^2}{2mL^2} \right).
\]

The oscillation period \( T = \frac{2\pi \hbar}{E_2 - E_1} \) follows from the \( \cos \left( \frac{(E_2 - E_1) t \hbar}{\pi} \right) \) dependence that results when you evaluate \( \psi^\star(x,t) \psi(x,t) \). So, \( T = 4mL^2 / 3\pi \hbar \). Therefore, \( T \Delta E = \frac{\sqrt{3}}{2} \pi \hbar \), which satisfies the energy-time version of the uncertainty principle.

5. For the potential step centered on zero, with \( E > V_0 \).

For \( x < 0 \), \( \psi(x) \) has an incident part travelling toward the right and a reflected part travelling to the left. The time-independent part of the wavefunction is

\[
\psi(x) = e^{ikx} + B e^{-ikx} \quad k = \sqrt{\frac{2mE}{\hbar}}, \text{ where for convenience we take a coefficient of 1 for the incident part.}
\]

For \( x > 0 \), \( \psi(x) \) consists of a transmitted part (toward the right)
\[ \psi(x) = Ce^{ikx} \quad k' = \frac{\sqrt{2m(E-V_0)}}{\hbar}. \]

The boundary conditions at \( x = 0 \) yields

\[ 1 + B = C \]

and

\[ ik - ikB = ik'C. \]

Solving these equations,

\[ B = \frac{k - k'}{k + k'} \quad C = \frac{2k}{k + k'}. \]

The flux is defined as

\[ j(x) = \frac{\hbar}{2im} \left[ \psi^*(x) \frac{\partial \psi(x)}{\partial x} - \psi(x) \frac{\partial \psi^*(x)}{\partial x} \right]. \]

Using the wavefunction and its derivative gives

\[ j(x) = \frac{\hbar}{2mi} \left[ (e^{-ikx} + B*e^{ikx})\left(ik e^{ikx} - ikB e^{-ikx}\right) - (e^{ikx} + B e^{-ikx})(-ik'e^{-ik'x} + ik'B*e^{ik'x}) \right] = \frac{\hbar k}{m} \left(1 - |B|^2\right). \]

\[ (x < 0) \]

\[ j(x) = \frac{\hbar k'}{m} |C|^2 \quad (x > 0) \]

The sum of the reflected \( \frac{\hbar k}{m} |B|^2 \) and transmitted \( \frac{\hbar k'}{m} |C|^2 \) waves is

\[ \frac{\hbar k}{m} |B|^2 + \frac{\hbar k'}{m} |C|^2 = \frac{\hbar}{m} \left[ \frac{k(k-k')^2 + k'(2k)^2}{(k+k')^2} \right] = \frac{\hbar k}{m} [1], \]

matching the incident probability current.

The reflection probability \( P_R \) is the ratio of the reflected and incident probability currents (the incident is just \( k \)): 

\[ \frac{\hbar k}{m} |B|^2 \]

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\[ P_R = \frac{(\hbar k/m)B^2}{(\hbar k/m)} = B^2 = \frac{(k-k')^2}{(k+k')^2}. \]

The corresponding transmission probability \( P_T \) is

\[ P_T = \frac{(\hbar k'/m)C^2}{(\hbar k/m)} = \frac{k'\left(\frac{2k}{k+k'}\right)^2}{k+k'} = \frac{4kk'}{(k+k')^2}. \]

\[ P_R + P_T = \frac{(k-k')^2 + 4kk'}{(k+k')^2} = \frac{(k+k')^2}{(k+k')^2} = 1. \]

6. Since the square well is a barrier of negative height, we can obtain \( T \) by replacing \( V_0 \) by \(-V_0\) in \( K^2 = k^2 - k_0^2 \)

(a) Then
\[ K^2 = k^2 - k_0^2 = (2m/\hbar)(V_0 - E) \rightarrow (2m/\hbar)(-V_0 - E) = -\kappa^2. \]
Noting that \( \sinh(\kappa) = \sinh(iK) = i\sin(K) \) we get 23.10.

(b) Therefore, when \( 2Ka = n\pi \), where \( \sin 2Ka = 0 \), we have \( T = 1 \). That is, the transmission probability is one, and the barrier appears transparent!!!!!

(c) When \( E >> V_0 \), \( K \approx \sqrt{2mE/\hbar} = k \), and the transmitted wave amplitude becomes

\[ T = \frac{1}{1 + \frac{V_0}{E}\sin^2(2ka)} \rightarrow 1. \]

That is, for \( E >> V_0 \), the particle doesn’t see the potential well.