I. (48 points)

A. Do all wavefunctions vanish at infinity? If not, give an example of one that does not.

Clearly not. We used plane waves, these do not vanish at infinity.

B. Show that the potential energy is a Hermitian operator.

\[
\left( \int dx \phi_n^*(x) V(x) \phi_m(x) \right)^* = \int dx \phi_n^*(x) V(x) \phi_m(x) \quad \text{since} \quad V(x) = (V(x))^*
\]

C. For a potential barrier of the form

\[
V(x) = \begin{cases} 
V_0 & \text{for } x < L \\
0 & \text{for } x > L 
\end{cases} \quad (V_0 > 0)
\]

indicate the general nature of the wavefunctions in the two space ranges when \( E < V_0 \).

Inside the barrier region \( |x| < L \) the solutions must be decaying, so a linear combo of \( \exp(\pm kx) \) function. Outside, a linear combo of oscillations, \( \exp(\pm i\kappa x) \)
D. For a wave function $\Psi(x) = \sum a_n \varphi_n(x)$ how would you determine the coefficients $a_n$?

$$a_n = \int dx \phi_n^*(x) \Psi(x)$$

E. What is the orthonormality relation? What does it express in words?

$$\int dx \phi_n^*(x) \phi_m(x) = \delta_{nm}$$

Orthonormality is overlap between two wave functions. Giving zero when they are different and one when they are the same function.

F. Discuss in qualitative terms (you don’t need a formula) what happens to the width $a(t)$ of a wave packet as it develops in time from some initial width $a(0)$. How will $a(t)$ depend on particle mass?

The width of the packet must increase with time. For Gaussian wavepackets it increase linearly with time. Proportional to inverse of particle mass – electrons spread; baseballs do too, but takes a long time.

G. What is the definition of a Hermitian operator? Write it out in terms of Dirac notation.

An operator for which $\langle n | A | m \rangle^* = \langle m | A^\dagger | n \rangle = \langle m | A | n \rangle$ is true.

H. Classify the following possible wave functions as even or odd or neither in $x$ when defined on a symmetric interval: $\sin(x)$, $\cos(x)$, $\exp(-x)$.

Respectively: odd, even, neither.
II. (27 points)

The time dependent Schrödinger equation is

\[ i\hbar \frac{\partial \Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{d^2 \Psi(x,t)}{dx^2} \]

A. Use the method of separation of variables to write separate space and time equations.

B. Solve the time dependent one. Be careful to distinguish between the Hamiltonian operator

\[ H(x) = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \]

and the energy \( E = \hbar \omega \).

C. Show that the probability density is time independent.

Use \( \Psi(x,t) = \psi(x) g(t) \) and substitute in to get

\[ -\frac{\hbar^2}{2m} \frac{\psi''(x)}{\psi(x)} = -\frac{\hbar}{i} \frac{\dot{g}(t)}{g(t)} = \hbar \omega \]

The above must be separate equations equal to the separation constant \( \hbar \omega \):

\[ -\frac{\hbar^2}{2m} \frac{\psi''(x)}{\psi(x)} = \hbar \omega \quad \text{and} \quad -\frac{\hbar}{i} \frac{\dot{g}(t)}{g(t)} = \hbar \omega \]

Solution of \[ -\frac{\hbar}{i} \frac{\dot{g}(t)}{g(t)} = \hbar \omega \] as a first order in time equation is

\[ g(t) = e^{-i\omega t} g(0) \]

The probability density is obtained as

\[ \Psi^*(x,t) \Psi(x,t) = \psi^*(x) \psi(x) g^*(t) g(t) = \psi^*(x) \psi(x) g^*(0) g(0) \] and is therefore time independent.
III. (25 points)

The quantum equation of continuity states that \( \frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0 \) where \( \rho(x,t) = \psi^*(x,t)\psi(x,t) \) and \( j(x,t) = \frac{\hbar}{2mi} \left[ \psi^*(x,t) \frac{\partial \psi(x,t)}{\partial x} - \psi(x,t) \frac{\partial \psi^*(x,t)}{\partial x} \right] \).

Show that \( m \frac{\partial \langle x \rangle}{\partial t} = \langle p \rangle \) where \( \langle p \rangle \) is the expectation value of the momentum operator. (It is useful to integrate by parts in both terms of the continuity equation after using it to get the expectation values.)

Multiply the continuity equation by \( mx \) and integrate, over \( x \).

This gives

\[
m \int dx \frac{\partial \rho}{\partial t} = m \frac{\partial \langle x \rangle}{\partial t}
\]

where \( \langle x \rangle \) denotes the expectation value.

The right hand side, \( -\frac{\partial j}{\partial x} \), can be written as

\[
-m \int dx \frac{\partial j}{\partial x} = m \int dx j(x) \text{ where we integrated by parts.}
\]

\[
= m \frac{\hbar}{2im} \int \left( dx \psi^* (x) \frac{d\psi (x,t)}{dx} - \psi (x) \frac{d\psi^* (x,t)}{dx} \right) = \frac{\hbar}{i} \int dx \psi^* (x,t) \frac{d\psi (x,t)}{dx}
\]

where we integrated by parts on the second term in the first equality.

Therefore, \( \langle p \rangle (t) = \frac{\hbar}{i} \int dx \psi^* (x,t) \frac{d\psi (x,t)}{dx} = \int dx \psi^* (x,t) \hat{p}\psi (x,t) \)

Using the definition of the momentum operator \( \hat{p} = \frac{\hbar}{i} \frac{d}{dx} \)