week 8 HO by Ladder, from PC $H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2 = \frac{1}{2m}p^2 + \frac{1}{2}kx^2$

$k = m\omega^2$ (k is force const).

We are required to find state vectors ψ_E, such that

$$\mathsf{H}\psi_E = E\psi_E, \tag{9–256}$$

the operators x and p being subject to the commutation rules

$$[\mathsf{x}, \mathsf{p}] = i\hbar\mathsf{1}. \tag{9–257}$$

For simplicity, we introduce units of mass, length and time such that $m = k = \hbar = 1$. Then we have

$$\mathsf{H} = \tfrac{1}{2}(\mathsf{p}^2 + \mathsf{x}^2), \tag{9–258}$$

and

$$[\mathsf{x}, \mathsf{p}] = i\mathsf{1}. \tag{9–259}$$

It is useful to introduce, in addition to x and p, the operators[1]

$$\mathsf{a} = \frac{i}{\sqrt{2}}(\mathsf{p} - i\mathsf{x}), \qquad \mathsf{a}^\dagger = \frac{1}{\sqrt{2}i}(\mathsf{p} + i\mathsf{x}), \tag{9–260}$$

in terms of which

$H = aa^\dagger - \frac{1}{2}\mathbb{1}$

$$\mathsf{aa}^\dagger = \tfrac{1}{2}[\mathsf{p}^2 - i(\mathsf{xp} - \mathsf{px}) + \mathsf{x}^2] = \mathsf{H} + \tfrac{1}{2}\mathsf{1}, \tag{9–261}$$

$$\mathsf{a}^\dagger\mathsf{a} = \tfrac{1}{2}[\mathsf{p}^2 + i(\mathsf{xp} - \mathsf{px}) + \mathsf{x}^2] = \mathsf{H} - \tfrac{1}{2}\mathsf{1}. \tag{9–262}$$

It follows that the commutator of a and $\mathsf{a}^\dagger$ is

$$[\mathsf{a}, \mathsf{a}^\dagger] = \mathsf{1}, \tag{9–263}$$

and

$$\mathsf{H} = \tfrac{1}{2}(\mathsf{aa}^\dagger + \mathsf{a}^\dagger\mathsf{a}). \tag{9–264}$$

$aH = a[aa^\dagger - \frac{1}{2}\mathbb{1}]$

$\cancel{Ha = a[a^\dagger a}$

$Ha = (aa^\dagger - \frac{1}{2}\mathbb{1})a$

Also, it is easily verified that $[a,H] = aH - Ha = aaa^\dagger - \frac{1}{2}a - aa^\dagger a + \frac{1}{2}a = a(aa^\dagger) - a(a^\dagger a) = a[a,a^\dagger] = a$

$$[\mathsf{a}, \mathsf{H}] = \mathsf{a}, \qquad [\mathsf{a}^\dagger, \mathsf{H}] = -\mathsf{a}^\dagger. \tag{9–265}$$

From the definition of a scalar product, it follows that

$$(\psi_E, \mathsf{a}^\dagger\mathsf{a}\psi_E) = (\mathsf{a}\psi_E, \mathsf{a}\psi_E) \geq 0. \tag{9–266}$$

If we suppose that ψ_E is an eigenvector of H, then, by Eqs. (9–262) and (9–256),

$$(\psi_E, \mathsf{a}^\dagger\mathsf{a}\psi_E) = (\psi_E, (\mathsf{H} - \tfrac{1}{2})\psi_E) = (E - \tfrac{1}{2})(\psi_E, \psi_E), \tag{9–267}$$

from which it follows that

$$E \geq \tfrac{1}{2}. \tag{9–268}$$

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[1] $\mathsf{a}^\dagger$ is the Hermitian conjugate of a, because p and x are Hermitian.

Applying the operator a to each member of Eq. (9–256) and using Eq. (9–265), we obtain

$$\mathsf{aH}\psi_E = (\mathsf{Ha} + \mathsf{a})\psi_E = E\mathsf{a}\psi_E, \tag{9–269}$$

or

$$\mathsf{Ha}\psi_E = (E - 1)\mathsf{a}\psi_E. \tag{9–270}$$

The vector $\mathsf{a}\psi_E$ is an eigenvector of H belonging to the eigenvalue $E - 1$, that is, the *lowering operator* a generates the vector $\psi_{E-1} = \mathsf{a}\psi_E$ from the vector ψ_E. Repeated application of a therefore lowers E indefinitely, so long as none of the ψ is zero:

$$\mathsf{a}\psi_E = \psi_{E-1}, \qquad \mathsf{a}^2\psi_E = \psi_{E-2}, \qquad \mathsf{a}^3\psi_E = \psi_{E-3}, \qquad \text{etc.} \tag{9–271}$$

However, this result contradicts (9–268), unless the operation a produces the zero vector at some stage. Hence, there is an eigenvalue $E_0 = E - n$ for which $\psi_{E_0} \neq 0$, but

$$\mathsf{a}\psi_{E_0} = 0. \tag{9–272}$$

For this eigenvalue we have, by Eq. (9–262),

$$\mathsf{a}^\dagger\mathsf{a}\psi_{E_0} = (\mathsf{H} - \tfrac{1}{2}\mathsf{1})\psi_{E_0} = (E_0 - \tfrac{1}{2})\psi_{E_0} = 0, \tag{9–273}$$

whence, since ψ_{E_0} is not zero,

$$E_0 = \tfrac{1}{2}. \tag{9–274}$$

The smallest eigenvalue of H is $\frac{1}{2}$; therefore,

$$E = n + \tfrac{1}{2}, \qquad n = 0, 1, 2, \ldots, \tag{9–275}$$

which is exactly the result (5–114), obtained by working in the x-representation.

We now relabel the eigenfunctions ψ_E with the index n:

$$\psi_{E=n+1/2} = \text{constant} \times \psi_n, \tag{9–276}$$

$$\mathsf{H}\psi_n = (n + \tfrac{1}{2})\psi_n. \tag{9–277}$$

The algebraic properties of these eigenvectors follow easily from the properties of the matrices which represent a and $\mathsf{a}^\dagger$ in the energy representation. If we assume that the states ψ_n are normalized, so that

$$(\psi_{n'}, \psi_n) = \delta_{nn'}, \tag{9–278}$$

then we have

$$\mathsf{a}\psi_n = \alpha_n\psi_{n-1}, \qquad \mathsf{a}^\dagger\psi_n = \beta_n\psi_{n+1}. \tag{9–279}$$

$n'=1\ n=2\ \langle 1|a|2\rangle = \alpha_2\,\delta_{1,1} = \alpha_2$

$n'=2\ n=1\cdot \langle 2|a^\dagger|1\rangle = \beta_1\,\delta_{2,2} = \beta_1$

BUT $\langle n'|a^\dagger|n\rangle = (\langle n|a|n'\rangle)^*$ or $a^\dagger_{n'n} = a^*_{nn'}$

The second of these equations is a consequence of Eq. (9–261). The matrix elements of a and $\mathsf{a}^\dagger$ are therefore

$$(n'|\mathsf{a}|n) = \alpha_n\,\delta_{n',n-1}, \tag{9–280}$$

$$(n'|\mathsf{a}^\dagger|n) = \beta_n\,\delta_{n',n+1}, \tag{9–281}$$

whence the constants α_n and β_n are related by

$$\alpha_{n+1} = \beta_n^*. \tag{9–282}$$

Moreover, from Eqs. (9–261) and (9–277), the matrix element of $\mathsf{a}\mathsf{a}^\dagger$ is

$aa^\dagger = H + \frac{1}{2}\mathbb{1}$

$$(n'|\mathsf{a}\mathsf{a}^\dagger|n) = (n+1)\,\delta_{nn'}. \tag{9–283}$$

$n'=2\ n=1$

$\beta_1 = a^\dagger_{21} = a^*_{12} = \alpha_2^*$ or $\alpha_2 = \beta_1^*$

Writing out the matrix product, we obtain

$$(n'|\mathsf{a}\mathsf{a}^\dagger|n) = \sum_{n''} (n'|\mathsf{a}|n'')(n''|\mathsf{a}^\dagger|n)$$

$$= \sum_{n''} \alpha_{n''}\,\delta_{n'n''-1}\beta_n\,\delta_{n'',n+1} = \alpha_{n+1}\beta_n\,\delta_{nn'}, \tag{9–284}$$

whence

$$\alpha_{n+1}\beta_n = n+1, \tag{9–285}$$

and, by (9–282),

$$|\alpha_{n+1}|^2 = |\beta_n|^2 = n+1. \tag{9–286}$$

Hence, the arbitrary phases of the vectors ψ_n can be chosen so that

$$\alpha_n = \sqrt{n}, \qquad \beta_n = \sqrt{n+1}, \tag{9–287}$$

and we have

$$\mathsf{a}\psi_n = \sqrt{n}\,\psi_{n-1}, \qquad \mathsf{a}^\dagger\psi_n = \sqrt{n+1}\,\psi_{n+1}. \tag{9–288}$$

The relations

$$\mathsf{a} + \mathsf{a}^\dagger = \sqrt{2}\mathsf{x}, \qquad \mathsf{a} - \mathsf{a}^\dagger = \sqrt{2}i\mathsf{p} \tag{9–289}$$

therefore lead to

$$\sqrt{n}\,\psi_{n-1} - \sqrt{2}\mathsf{x}\psi_n + \sqrt{n+1}\,\psi_{n+1} = 0, \tag{9–290}$$

$\Leftarrow$ S. eq. and useful for selection rules for rad.-matter interactions

$$\sqrt{n}\,\psi_{n-1} - \sqrt{2}\,i\mathsf{p}\psi_n - \sqrt{n+1}\,\psi_{n+1} = 0. \tag{9–291}$$

These recurrence relations, which were also derived in Section 5–10, lead immediately to the Schrödinger equation, and we know that in the x-representation, the vector ψ_n has the components

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!\,\sqrt{\pi}}}\,e^{-(x^2/2)}H_n(x). \tag{9–292}$$

However, all the essential properties of the state vectors have been determined here in a manner which is independent of this representation.

The lowering and raising operators $\mathfrak{a}$ and $\mathfrak{a}^\dagger$ play a fundamental role in the quantum theory of the electromagnetic field. It is well known that the Maxwell field can be represented as a linear combination of harmonic oscillators of different frequencies in various states of excitation.[1] Consequently, the electromagnetic field can be represented by a Hamiltonian function which describes an infinite set of harmonic oscillators, each of which represents a normal mode of the field oscillations (cf. Section 1–3). In quantum theory, each of these oscillators is "quantized" by imposing the condition (9–257) upon each of the pairs of conjugate coordinates in the Hamiltonian. A stationary state of the system can then be described by giving the quantum number n for each of the normal-mode oscillators. Thus, the total energy of the field is a sum of terms of the form $(n + \frac{1}{2})\hbar\omega$; there is one such term for each normal mode. In a transition in which a quantum of energy $\hbar\omega$ is absorbed by an atomic system, the corresponding quantum number n is reduced by unity. Similarly, the addition of a photon to the field, by emission of a quantum, corresponds to an increase of n by one. Now, according to Eqs. (9–279), the operators $\mathfrak{a}$ and $\mathfrak{a}^\dagger$ have exactly this effect, and it is not surprising that, in the general theory, these operators appear as factors in the terms of the Hamiltonian of the system (atom + radiation field) which describe the interaction between the field and the charged particles. Indeed, it is this interaction which causes the absorption and emission of photons. When the operators $\mathfrak{a}$ and $\mathfrak{a}^\dagger$ are used in this manner, they are called *destruction* and *creation operators* for photons. The essential properties of $\mathfrak{a}$ and $\mathfrak{a}^\dagger$, which are required for the radiation theory, are expressed by the equations

$$(n'|\mathfrak{a}|n) = \sqrt{n}\,\delta_{n'n-1}, \qquad (n'|\mathfrak{a}^\dagger|n) = \sqrt{n+1}\,\delta_{n'n+1}. \tag{9–293}$$

The matrices which represent these operators are therefore:

$$a = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & 0 & \\ 0 & 0 & 0 & \sqrt{3} & 0 & \\ 0 & 0 & 0 & 0 & \sqrt{4} & \\ \vdots & & & & & \end{pmatrix} \qquad a^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & 0 & 0 & \\ 0 & \sqrt{2} & 0 & 0 & \\ 0 & 0 & \sqrt{3} & 0 & \\ \vdots & & & & \end{pmatrix}; \tag{9–294}$$

[1] L. Landau and E. Lifshitz, *The Classical Theory of Fields*. Reading, Mass.: Addison-Wesley Publishing Co., Inc., 1951, Chapter 6.

and, by Eqs. (9–289), x and p are the Hermitian matrices

$$x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \sqrt{1} & 0 & \cdots & \\ \sqrt{1} & 0 & \sqrt{2} & & \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} \\ \vdots & & & & \end{pmatrix},$$

$$p = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i\sqrt{1} & 0 & \cdots & \\ i\sqrt{1} & 0 & -i\sqrt{2} & & \\ 0 & i\sqrt{2} & 0 & -i\sqrt{3} & \\ 0 & 0 & i\sqrt{3} & 0 & -i\sqrt{4} \\ \vdots & & & & \end{pmatrix}. \tag{9–295}$$

The Hamiltonian **H** is represented by the diagonal matrix

$$H = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \cdots \\ 0 & \frac{3}{2} & 0 & \\ 0 & 0 & \frac{5}{2} & \\ \vdots & & & \end{pmatrix}. \tag{9–296}$$

Note that the selection rules for the matrix elements of x and p, namely

$$\Delta n = \pm 1,$$

appear explicitly in Eq. (9–295), in that nonzero elements occur only in the two diagonals adjacent to the principal diagonal.

References

Born, M., and Jordan, P., *Elementare Quantenmechanik.* Berlin: Julius Springer, 1930, Chapter 2.

Courant, R., and Hilbert, D., *Methods of Mathematical Physics.* New York: Interscience Publishers, Inc., 1953. Chapter 1 describes the algebra of linear transformations in a form immediately adaptable to the solution of physical problems.

Halmos, P. R., *Finite Dimensional Vector Spaces.* Princeton: Princeton University Press, 1942. A thorough and readable discussion of the mathematical concepts used in this chapter.

TO GENERATE EXPLICIT SOLUTIONS

$[x,p] = i\hbar$

$p = \frac{\hbar}{i}\frac{d}{dx}$

$a^\dagger \psi_n = \sqrt{n+1}\,\psi_{n+1}$

$$\psi_{n+1} = \frac{a^\dagger}{\sqrt{n+1}}\psi_n$$

Start at $n=0$ $\quad \psi_1 = \frac{a^\dagger}{\sqrt{1}}\psi_0 \quad \psi_2 = \frac{a^\dagger}{\sqrt{2}}\psi_1 \ldots$

$$\boxed{\psi_n = \frac{(a^\dagger)^n}{\sqrt{n!}}\psi_0}$$

then $\quad a = \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx}\right) \qquad a^\dagger = \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right)$

$a\psi_{E_0} = 0 \qquad \frac{i}{\sqrt{2}}(p - ix)\,\psi_0(x) \qquad \left(a^\dagger = \frac{1}{\sqrt{2}}\frac{1}{i}(p+ix)\right)$

$$= \frac{1}{\sqrt{2}}\left(+\frac{d}{dx} + x\right)\psi_0(x) = 0 \qquad \left(\frac{d}{dx} + x\right)\psi_0(x) = 0$$

$$\psi(x) = A\,e^{-\frac{x^2}{2}}$$

$$\int dx\, |\psi(x)|^2 = A^2 \int_{-\infty}^{+\infty} dx \left(e^{-x^2/2}\right)^2 \qquad A^2 = \frac{1}{\sqrt{\pi}}$$

$$\psi_0(x) = \frac{1}{(\sqrt{\pi})^{1/2}}\, e^{-x^2/2}$$

$$\psi_n(x) = \frac{1}{\sqrt{n!}}\left[\frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right)\right]^n \frac{1}{(\sqrt{\pi})^{1/2}}\, e^{-x^2/2}$$

But $\quad \left(x - \frac{d}{dx}\right)^n e^{-x^2/2} = (-1)^n e^{x^2/2}\left(\frac{d}{dx}\right)^n e^{-x^2}$

$\psi_n(x) = A_n\, e^{-x^2/2}\, H_n(x) \qquad H_n(x)$ are Hermite polys —

$H_0 = 1 \quad H_1 = 2x \quad H_2 = 4x^2 - 2 \quad H_3 = 8x^3 - 12x, \ldots$

When this operator acts on the energy eigenfunctions $\psi_{E_n}(x)$, the result is

$$P\psi_{E_n}(x) = \psi_{E_n}(-x) = (-1)^n \psi_{E_n}(x) \tag{41}$$

Thus, $\psi_{E_n}(x)$ is an eigenfunction of P with eigenvalue $(-1)^n$.

The harmonic-oscillator wavefunctions $\psi_{E_n}(x)$ form a complete orthonormal set. The completeness relation for these wavefunctions is

$$\begin{aligned}\delta(x - x') &= \sum_n \psi_{E_n}(x)\psi_{E_n}(x') \\ &= \sum_n \frac{1}{2^n}\frac{1}{n!}\frac{1}{x_0\sqrt{\pi}}\, e^{-\frac{1}{2}x^2/x_0^2}H_n(x/x_0)e^{-\frac{1}{2}x'^2/x_0^2}H_n(x'/x_0)\end{aligned} \tag{42}$$

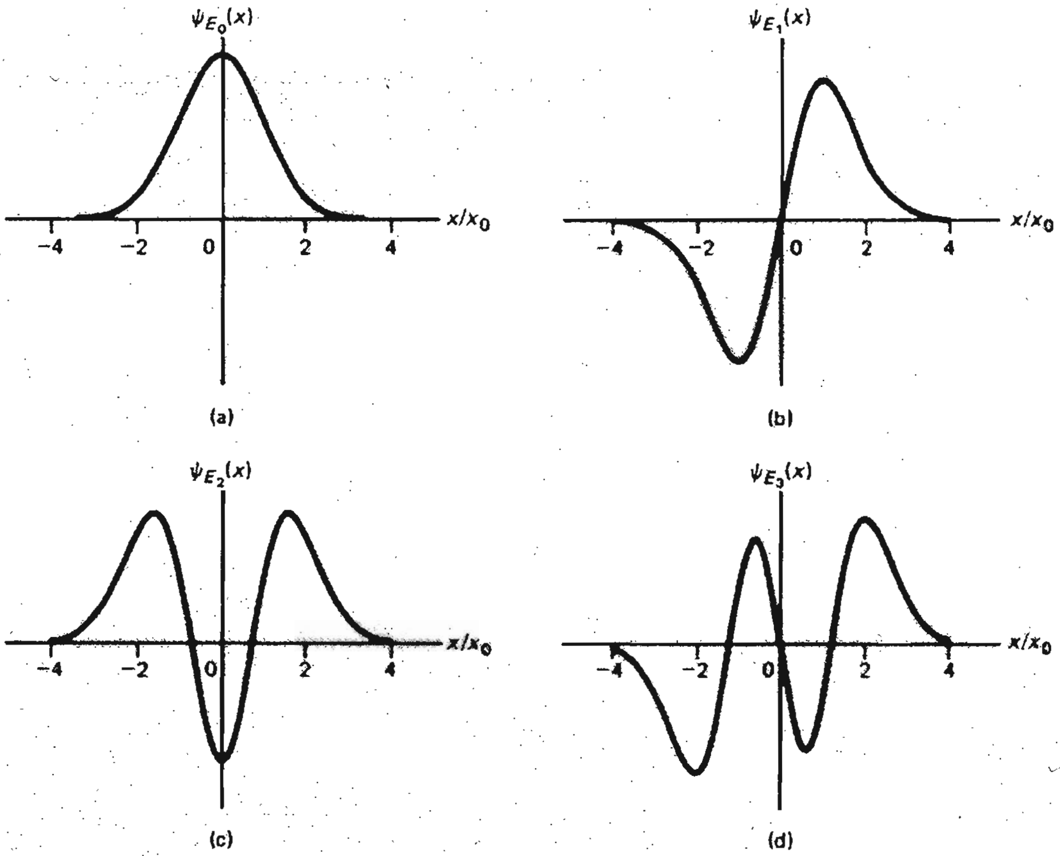
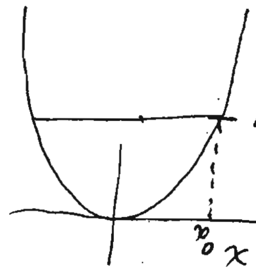


Fig. 6.2 The energy eigenfunctions ψ_{E_0}, ψ_{E_1}, ψ_{E_2}, and ψ_{E_3} for the harmonic oscillator.

Classical Connection for HO

For a classical oscillator

$$E = \tfrac{1}{2} m v^2 + \tfrac{1}{2} k x^2$$



$V(x) = \tfrac{1}{2} k x^2$

$E = \tfrac{1}{2} k x_0^2$ that is x_0 is the "turning point" of the motion where the total energy E is all ~~kin~~ potential energy.

$v(x_0) = v(-x_0) = 0$

Its useful to think about $v(x)$, defined viz

$$E = \tfrac{1}{2} m v^2(x) + \tfrac{1}{2} k x^2 \equiv \tfrac{1}{2} k x_0^2$$

So

$$\tfrac{1}{2} m v^2(x) = \tfrac{1}{2} k (x_0^2 - x^2)$$

$$v(x) = \sqrt{k/m}\left(\sqrt{x_0^2 - x^2}\right) = \omega \sqrt{x_0^2 - x^2}$$

Consider the probability that the (classical) oscillator finds itself between x, $x + dx$

$P(x)\,dx \equiv \frac{dt}{T}$ where $\frac{dt}{T}$ is fraction of the time oscillator spends in the interval dx

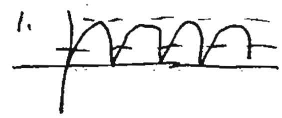
$T = 2\pi/\omega$ is period of oscillation $\quad dt = \overset{2 \text{ (either side)}}{\downarrow} dx/v(x)$

$$P(x)\,dx = \left(\frac{\omega}{2\pi}\right) \frac{\overset{2}{\downarrow} dx}{\omega \sqrt{x_0^2 - x^2}} = \frac{\overset{2}{\downarrow} dx}{2\pi \sqrt{x_0^2 - x^2}} = \frac{dx}{\sqrt{x_0^2 - x^2}}$$

For large n, the properties of the Hermite polynomials (asymptotic expansion in n — see e.g. A+S). yield

$$|\psi_n(x)|^2 = \frac{1}{2^n n! \sqrt{\pi}} e^{-x^2} H_n^2(x) \underset{n\gg 1}{\sim} \frac{2}{\pi\sqrt{2n-x^2}} \cos^2\left[\left(2n+\tfrac{1}{2}\right)\frac{x}{\sqrt{2n}} - \frac{n\pi}{2}\right]$$

The ave of $\cos^2$ over many periods is $1/2$.



$\cos^2$

And $2n \sim x_0^2$

So $$P(x)\,dx = |\psi_n(x)|^2 dx \sim \frac{1}{\pi\sqrt{x_0^2 - x^2}}$$

This is an example of the CORRESPONDENCE PRINCIPLE. That says for "large" values of the quantum number quantum mechanical predictions approach classical predictions.

? dimension? $2n \sim$ energy of osc. so something like $x_0^2 \sim E_{class}$.

Parity

The HO solutions are nondegenerate.

If we set $x \Rightarrow -x$ in S. eq.

$$-\frac{d^2}{dx^2} + x^2 \xrightarrow[x \to -x]{} -\frac{d^2}{dx^2} + x^2$$

So

$$\psi(-x) = c\,\psi(x) \qquad c \text{ some const.}$$

apply P (parity) operator

$$P\,\psi(+x) = \cancel{c\,\psi(x)} = c\,\psi(-x)$$

apply again $\quad P^2\psi = P(P\psi) = P\,c\,\psi(-x) = c^2\psi(x)$

$$P^2 = 1$$

$\Rightarrow \quad c^2 = \pm 1$.

Therefore $\quad \psi(x) = \pm\,\psi(-x)$

Solutions are either even $(+)$ or odd (-1)

From the Hermites it's clear

$$\psi_n(x) = (-1)^n\,\psi_n(-x)$$

i.e. $\quad n = 0, 2, \ldots$ are even

$\quad\quad n = 1, 3, \ldots$ are odd.

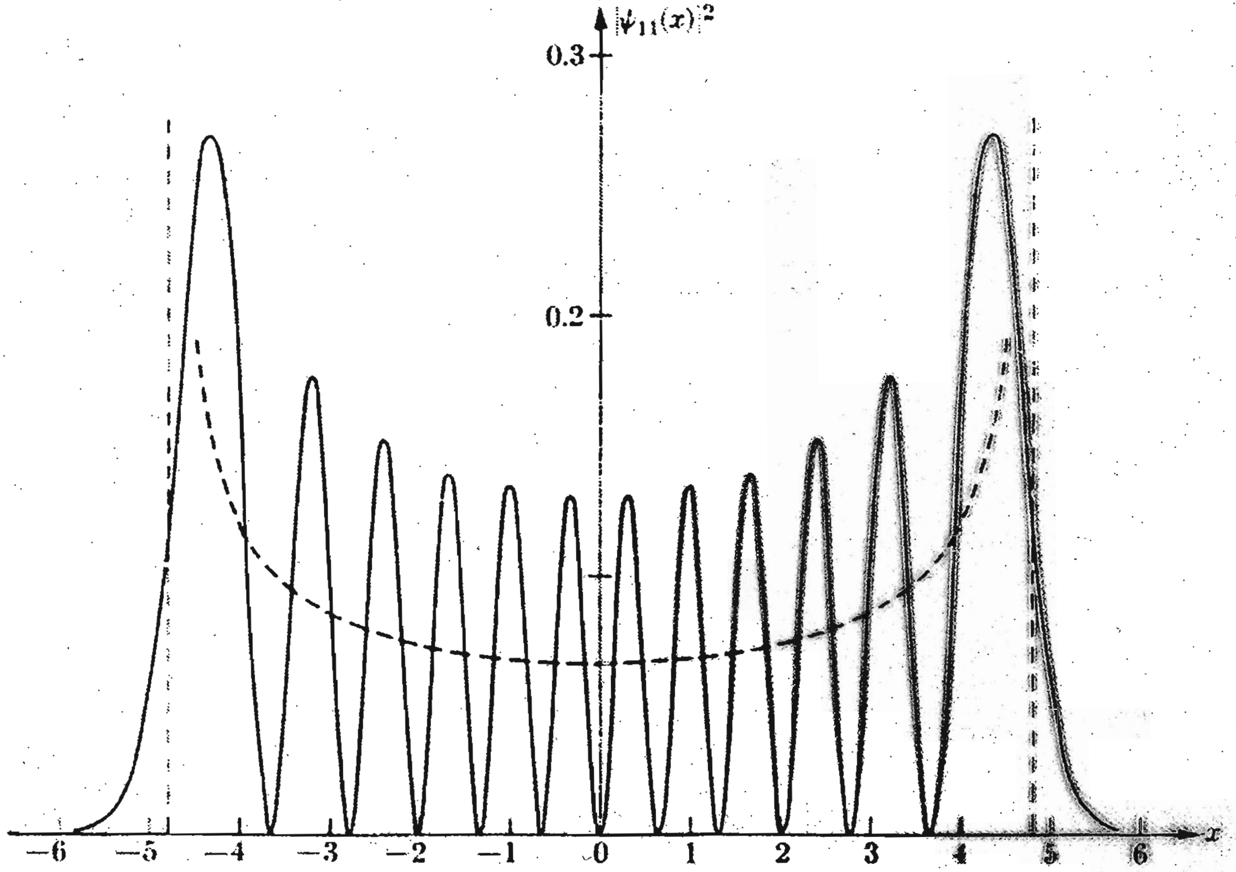


FIG. 5–20. Probability density for the harmonic oscillator in the state ψ_{11}. The broken curve represents the classical probability distribution, as given by Eq. (5–143). [Cf. J. B. Russell, "A Table of Hermite Functions," *J. Math. Phys.* **12**, 291 (1933), and E. R. Smith, "Zeros of the Hermitian Polynomials," *Am. Math. Monthly* **43**, 354 (1936).]